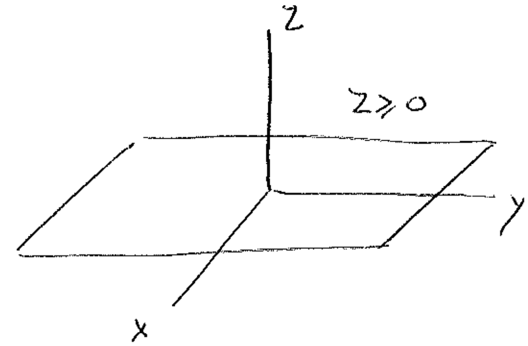
Problem 2.7 (only parts (a), (b))

(a) To address this problem we have to go back to Chapter 1 and the definition of Green functions.

Let us use (1.40):

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}')$$

with $\nabla'^2 F(\vec{x},\vec{x}') = 0$ inside the volume $z \geq 0$



+ Dirichlet boundary conditions

Remember that intuitively $\frac{1}{|\vec{x}-\vec{x}'|}$ is proportional to the potential created at $\vec{x}$ by a charge at $\vec{x}'$!

Then, it may be a good "guess" to assume the presence of an image charge at $\vec{x}''$ and see how it goes. For a plane, we know the image charge has same (x, y) coordinates but it is on the opposite side of plane. Thus, $\vec{x}'' = (x', y', -z')$ may work.

Since this "image charge" is outside $z \geq 0$, then

$$\nabla'^2 \left(\frac{1}{|\vec{x}-\vec{x}''|}\right) = 0 \quad z \geq 0 \quad \text{because } z'' = -z'.$$

$$\boxed{G_D(\vec{x},\vec{x}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}}$$

image charge is opposite in sign

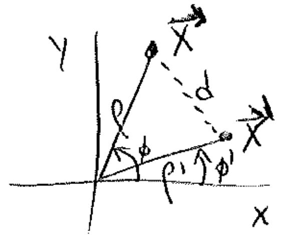
If $z=0$, then $G_D(\vec{x},\vec{x}')=0$ which is the boundary condition requested. So this is the Dirichlet Green function, thus the subindex "D".

As $|\vec{x}|\to\infty$, $G_D(\vec{x},\vec{x}')\longrightarrow 0$, so the condition at ∞ is also properly satisfied.

Since (b) will need cylindrical coordinates, then let us write $G_D(\vec{x},\vec{x}')$ in those coordinates:

$$G_D(\vec{x},\vec{x}') = \frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2}} - \frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z+z')^2}}$$

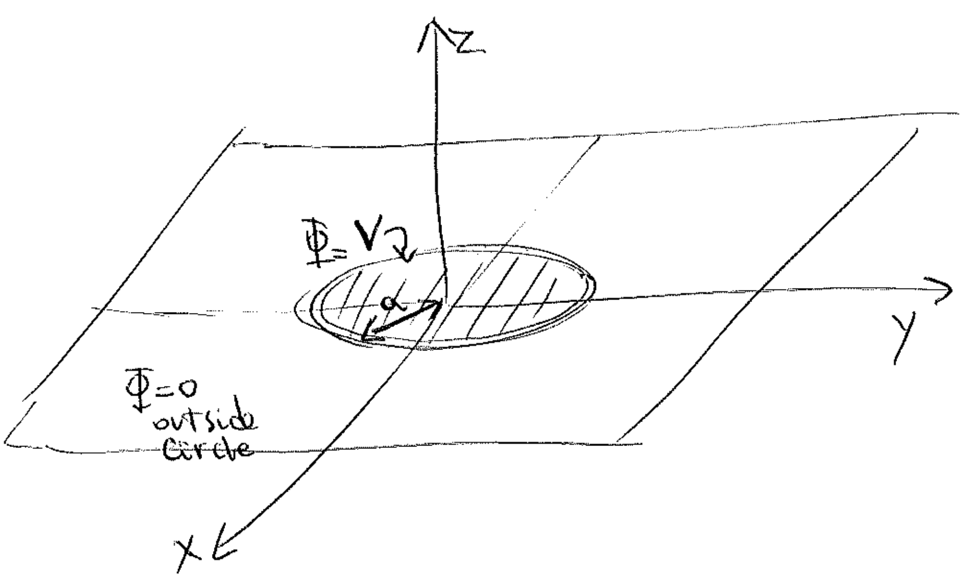
From "above" the x-y plane is:



$d^2=\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')$

cosine theorem

(b) Consider a potential Φ such that



We have $G_D(\vec{x}, \vec{x}')$, thus we can solve the problem using (1.44):

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{x}') G_D(\vec{x}, \vec{x}') \, d^3x' - \frac{1}{4\pi} \oint_S \Phi(\vec{x}') \frac{\partial G_D}{\partial n'} \, da'$$

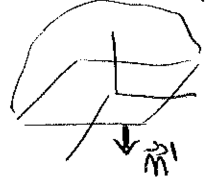
appears "strange" that Φ is part of the solution for Φ, but $\Phi(\vec{x}')$ in the second term is only needed at S, and the goal is to find it at any point in V.

In this problem, $\rho(\vec{x}') \equiv 0$ thus the first term does not contribute.
In $\oint_S$ in principle we should calculate the contribution of $S \to \infty$, but we know $G_D \to 0$ plus Φ should also $\to 0$ at large distances on physical grounds.

Then:

$$\Phi(\vec{x}) = -\frac{1}{4\pi} \int_{\substack{z'=0 \\ \text{plane}}} \Phi(\vec{x}') \left(-\frac{\partial G_D(\vec{x}, \vec{x}')}{\partial z'} \right) da'$$

Note that in $\frac{\partial G_D}{\partial n'}$, the unit vector $\vec{n}'$ points away from volume



Thus $\frac{\partial G_D}{\partial n'} = -\frac{\partial G_D}{\partial z'}$

With G_D in cylindrical coordinates we can calculate $\frac{\partial G_D}{\partial z'}$:

$$-\left.\frac{\partial G_D}{\partial z'}\right|_{z'=0} = -\Big\{\left(-\frac{1}{2}\right)\left[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2\right]^{-3/2} 2\overbrace{(z-z')}^{(z'-z)}$$

$$\left. -\left(-\frac{1}{2}\right)\left[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z+z')^2\right]^{-3/2} 2\,(z+z')\right|_{z'=0}\Big\}$$

$$= \frac{-2z}{\left[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2\right]^{3/2}}$$

Then:

$$\boxed{\Phi(\rho,\phi,z) = \frac{z}{2\pi}\int_0^{\infty}\int_0^{2\pi}\Phi'(\rho',\phi',0)\,\frac{\rho'\,d\rho'\,d\phi'}{\left(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2\right)^{3/2}}}$$

$$\boxed{= \frac{zV}{2\pi}\int_0^{2\pi}d\phi'\int_0^{a}\rho'\,d\rho'\cdot\frac{1}{\left[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2\right]^{3/2}}}$$

Since this problem has azimuthal symmetry the result cannot depend on ϕ. This can be easily seen by merely defining $\phi'' = -\phi + \phi'$

$$\cos(\phi' - \phi) = \cos(\phi - \phi') = \cos\phi''$$

$$d\phi'' = d\phi'$$

Then:

$$\boxed{\Phi(\rho, z) = \frac{zV}{2\pi} \int_0^{2\pi} d\phi' \int_0^{a} \rho' d\rho' \cdot \frac{1}{(\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi' + z^2)^{3/2}}}$$