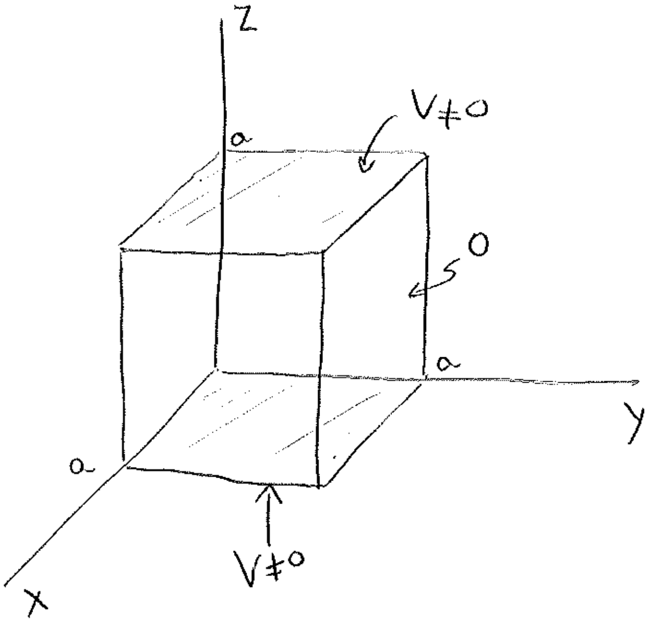
Problem 2.23 (only part(a))



Following Sec. 2.9 of the book, the general solution involves combinations of sines and cosines as in (2.52). But here we want Φ to be zero at $x=0$ and a and $y=0$ and a. Then, we must choose the sine solutions, as in the example of Sec. 2.9, for x and y.

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} \left(A_{nm}\, e^{\gamma_{nm} z} + B_{nm}\, e^{-\gamma_{nm} z} \right) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right)$$

with regards to "z" we cannot use "sinh" as in the example of Sec. 2.9 because Φ does not cancel at $z=0$ as in that example. Thus we must keep it generic

with

$$\gamma_{nm} = \frac{\pi}{a}\sqrt{n^2+m^2}$$

0 at $x=0$, $y=0$
0 at $x=a$, $y=a$

Now we have to introduce the boundary condition.
Here we will use the trick of combining the potential for the cases $[\Phi(z=a)=V, \Phi(z=0)=0]$ with $[\Phi(z=a)=0, \Phi(z=0)=V]$

For the first case:

$$\Phi = \sum_{n,m} A'_{nm} \underbrace{\sinh(\gamma_{nm} z)}_{\substack{=0 \text{ at } z=0 \\ \text{which is what} \\ \text{we want}}} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right)$$

$$\Phi(x,y,a) = V = \sum_{n,m} A'_{nm} \sinh(\gamma_{nm} a) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right)$$

To find A'_{nm} let use orthonormality properties as in the example of Sec. 2.9.

$$V \int_0^a \sin\left(\frac{n'\pi x}{a}\right) dx \int_0^a \sin\left(\frac{m'\pi y}{a}\right) dy =$$

$$= \sum_{n,m} A'_{nm} \sinh(\gamma_{nm} a) \underbrace{\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n'\pi x}{a}\right) dx}_{\frac{a}{2}\delta_{nn'}} \underbrace{\int_0^a \sin\left(\frac{m\pi y}{a}\right) \sin\left(\frac{m'\pi y}{a}\right) dy}_{\frac{a}{2}\delta_{mm'}}$$

$$= A'_{n'm'} \sinh(\gamma_{n'm'} a) \frac{a^2}{4}$$

Switching $n'm' \to nm$ again:

$$A'_{nm} = \frac{1}{\frac{a^2}{4}\sinh(\gamma_{nm}a)} \cdot V\left(\frac{2a}{n\pi}\right)\left(\frac{2a}{m\pi}\right)$$

$$= \boxed{\frac{16V}{nm\pi^2 \sinh(\gamma_{nm}a)}}$$ if both n, m are odd.

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right)dx =$$

$$= -\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\Big|_0^a =$$

$$= \left(\frac{-a}{n\pi}\right)\left[(-1)^n - 1\right]$$

$$= \frac{a}{n\pi}\cdot\left[1-(-1)^n\right] = \begin{cases} 0 & n \text{ even} \\ \frac{2a}{n\pi} & n \text{ odd} \end{cases}$$

Now we have to repeat for $\Phi(z=a)=0$, $\Phi(z=0)=V$.

In this case we cannot use $\sinh(\gamma_{nm}z)$ but a linear combo of exponentials.

$$\Phi(x,y,z) = \sum_{n,m}\left(A_{nm}\, e^{\gamma_{nm}z} + B_{nm}\, e^{-\gamma_{nm}z}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)$$

$\Phi(x,y,a) = 0$ means $\left(A_{nm}e^{\gamma_{nm}a} + B_{nm}e^{-\gamma_{nm}a}\right) = 0$

or $B_{nm} = -A_{nm}e^{2\gamma_{nm}a}$

$$\Phi(x,y,0) = V = \sum_{n,m} A_{nm}\left(1 - e^{2\gamma_{nm}a}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)$$

We have to use orthonormality properties again:

$$\int_0^a\int_0^a V \sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{a}\right) dx\, dy =$$

$$= \sum_{n,m} A_{nm}\left(1-e^{2\gamma_{nm}a}\right) \underbrace{\int_0^a dx \sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)}_{\frac{a}{2}\delta_{nn'}} \underbrace{\int_0^a dy \sin\left(\frac{m'\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right)}_{\frac{a}{2}\delta_{mm'}}$$

$$= \sum_{n,m} \frac{a^2}{4}\delta_{nn'}\delta_{mm'} A_{nm}\left(1-e^{2\gamma_{nm}a}\right)$$

$$= \frac{a^2}{4} A_{n'm'}\left(1-e^{2\gamma_{n'm'}a}\right).$$

$$\to \frac{2a}{n'\pi}\cdot\frac{2a}{m'\pi}\cdot V \quad (n', m' \text{ odd as before})$$

$$\boxed{A_{nm} = \frac{16V}{nm\pi^2\left(1-e^{2\gamma_{nm}a}\right)}}, \quad (n, m = \text{odd})$$

Combining both solutions we get:

$$\boxed{\Phi(x,y,z) = \frac{16V}{\pi^2}\sum_{\substack{n,m\\ \text{odd}}} \frac{1}{nm}\left[\frac{\left(e^{\gamma_{nm}z} - e^{2\gamma_{nm}a}e^{-\gamma_{nm}z}\right)}{\left(1-e^{2\gamma_{nm}a}\right)} + \frac{\sinh(\gamma_{nm}z)}{\sinh(\gamma_{nm}a)}\right]\cdot \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)}$$