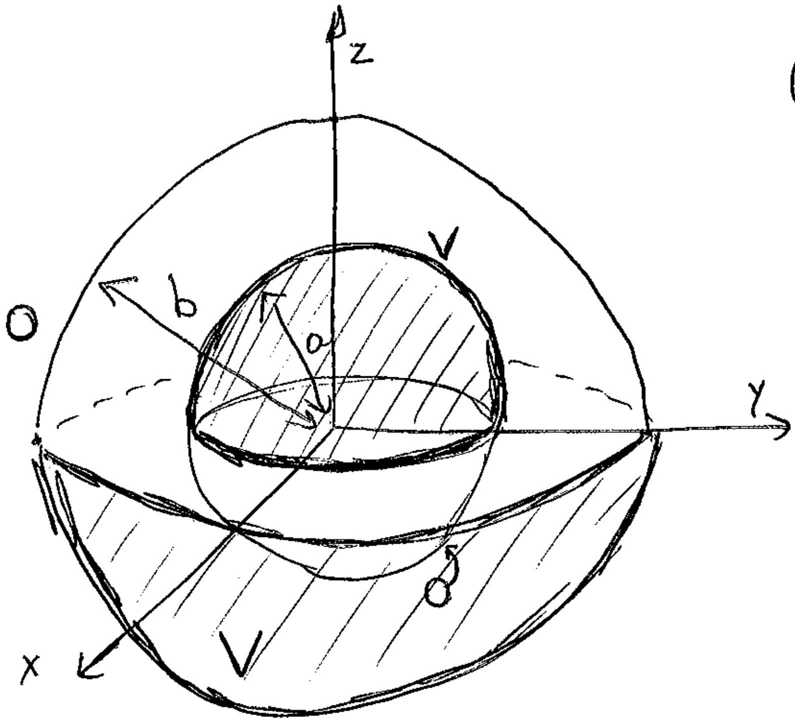

Problem 3.1

(without $b \to \infty$, $a \to 0$ checks)

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

(3.33)

for problems with azimuthal symmetry.

At $r = a$, we know that $\Phi(a,\theta) = \begin{cases} V & \text{if } \theta \in (0, \pi/2) \\ 0 & \text{if } \theta \in (\pi/2, \pi) \end{cases}$

In general for an arbitrary function $\Phi(a,\theta)$:

$$\Phi(a,\theta) = \sum_l \left(A_l a^l + \frac{B_l}{a^{l+1}} \right) P_l(\cos\theta)$$

$$\int_0^{\pi} \Phi(a,\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta = \sum_l \underbrace{\int_0^{\pi} d\theta \sin\theta \, P_{l'}(\cos\theta) P_l(\cos\theta)}_{\frac{2}{2l+1}\delta_{ll'} \ (3.21)} \left(A_l a^l + \frac{B_l}{a^{l+1}} \right)$$

now specialize to $\begin{cases} V, \theta \in (0, \pi/2) \\ 0, \theta \in (\pi/2, \pi) \end{cases}$

$$= \frac{2}{2l'+1} \left(A_{l'} a^{l'} + \frac{B_{l'}}{a^{l'+1}} \right)$$

and switch $l' \to l$:

$$V\int_0^{\pi/2} d\theta \sin\theta\, P_l(\cos\theta) = \frac{2}{2l+1}\left(A_l a^l + \frac{B_l}{a^{l+1}}\right)$$

Repeating now for $r=b$, it is obvious that

$$V\int_{\pi/2}^{\pi} d\theta \sin\theta\, P_l(\cos\theta) = \frac{2}{2l+1}\left(A_l b^l + \frac{B_l}{b^{l+1}}\right)$$

We must consider terms up to $l=4$. Using (3.15):

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2-1)$$
$$P_3(x) = \frac{1}{2}(5x^3-3x)$$
$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

$$\int_0^{\pi/2} d\theta \sin\theta = -\cos\theta\Big|_0^{\pi/2} = 1 \quad ; \quad \int_0^{\pi/2} d\theta \sin\theta\cos\theta = \frac{1}{2}\sin^2\theta\Big|_0^{\pi/2} = \frac{1}{2}$$

$$\int_0^{\pi/2} d\theta \sin\theta\, \frac{1}{2}(3\cos^2\theta - 1) = -\frac{3}{2}\frac{\cos^3\theta}{3}\Big|_0^{\pi/2} - \frac{1}{2}\cdot 1 = \frac{1}{2} - \frac{1}{2} = 0$$

$$\int_0^{\pi/2} d\theta \sin\theta\, \frac{1}{2}(5x^3-3x) = \frac{5}{2}\left(-\frac{\cos^4\theta}{4}\right)\Big|_0^{\pi/2} - \frac{3}{2}\cdot\frac{1}{2} = \frac{5}{2\cdot 4} - \frac{3}{4} = -\frac{1}{8}$$

$$\int_0^{\pi/2} d\theta \sin\theta \;\; \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) = \frac{35}{8}\left(-\frac{\cos^5\theta}{5}\right)\Bigg|_0^{\pi/2} - \frac{30}{8}\left(-\frac{\cos^3\theta}{3}\right)\Bigg|_0^{\pi/2} + \frac{3}{8}\cdot 1 =$$

$$= \frac{35}{8.5} - \frac{30}{8.3} + \frac{3}{8} = \frac{35.3 - 30.5 + 3.5.3}{8.5.3} = 0$$

Then, the first equation gives:

$$V = 2\left(A_0 + \frac{B_0}{a}\right) \qquad (\ell = 0)$$

$$V/2 = \frac{2}{3}\left(A_1 a + \frac{B_1}{a^2}\right) \qquad (\ell = 1)$$

$$0 = \frac{2}{5}\left(A_2 a^2 + \frac{B_2}{a^3}\right) \qquad (\ell = 2)$$

$$-\frac{V}{8} = \frac{2}{7}\left(A_3 a^3 + \frac{B_3}{a^4}\right) \qquad (\ell = 3)$$

$$0 = \frac{2}{9}\left(A_4 a^4 + \frac{B_4}{a^5}\right) \qquad (\ell = 4)$$

We must repeat now for the other integral:

$$\int_{\pi/2}^{\pi} d\theta \sin\theta = -\cos\theta\Big|_{\pi/2}^{\pi} = 1 \quad ; \int_{\pi/2}^{\pi} d\theta \sin\theta\cos\theta = -\frac{\cos^2\theta}{2}\Bigg|_{\pi/2}^{\pi} = -\frac{1}{2}$$

$$\int_{\pi/2}^{\pi} d\theta \sin\theta \;\frac{1}{2}\left(3\cos^2\theta - 1\right) = \frac{3}{2}\left(-\frac{\cos^3\theta}{3}\right)\Bigg|_{\pi/2}^{\pi} - \frac{1}{2}.1 = \frac{1}{2} - \frac{1}{2} = 0$$

$$\int_{\pi/2}^{\pi} d\theta \sin\theta \;\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right) = \frac{5}{2}\left(-\frac{\cos^4\theta}{4}\right)\Bigg|_{\pi/2}^{\pi} - \frac{3}{2}\left(-\frac{1}{2}\right) = -\frac{5}{8} + \frac{3}{4} = \frac{1}{8}$$

$$\int_{\pi/2}^{\pi} d\theta \sin\theta \;\frac{1}{8}\left(35\cos^4\theta - 30\cos^2\theta + 3\right) = \frac{35}{8}\left(-\frac{\cos^5\theta}{5}\right)\Bigg|_{\pi/2}^{\pi} - \frac{30}{8}\left(-\frac{\cos^3\theta}{3}\right)\Bigg|_{\pi/2}^{\pi} + \frac{3}{8}.1 = +\frac{35}{8.5} - \frac{30}{8.3} + \frac{3}{8} = 0$$

The second Φ equation gives:

$$V = 2\left(A_0 + \frac{B_0}{b}\right) \quad (l=0)$$

$$-\frac{V}{2} = \frac{2}{3}\left(A_1 b + \frac{B_1}{b^2}\right) \quad (l=1)$$

$$0 = \frac{2}{5}\left(A_2 b^2 + \frac{B_2}{b^3}\right) \quad (l=2)$$

$$\frac{V}{8} = \frac{2}{7}\left(A_3 b^3 + \frac{B_3}{b^4}\right) \quad (l=3)$$

$$0 = \frac{2}{9}\left(A_4 b^4 + \frac{B_4}{b^5}\right) \quad (l=4)$$

The Eqs. for $l=2$ are $A_2 + \frac{B_2}{a^5} = 0$ and $A_2 + \frac{B_2}{b^5} = 0$
which only have $\boxed{A_2 = B_2 = 0}$ as solution

The same for $l = 4$.

Then, only $l = 0, 1,$ and 2 contribute.

The $l=0$ Eqs. are $\frac{V}{2} = A_0 + \frac{B_0}{a}$; $\frac{V}{2} = A_0 + \frac{B_0}{b}$
which has as solution $\boxed{B_0 = 0, \ A_0 = V/2}$

The $l=1$ Eqs. are $\frac{3}{4} V = A_1 a + \frac{B_1}{a^2}$; $-\frac{3V}{4} = A_1 b + \frac{B_1}{b^2}$

$$\boxed{A_1 = \frac{-3V(b^2+a^2)}{4(b^3-a^3)} \ ; \ B_1 = \frac{+3Va^2b^2(b+a)}{4(b^3-a^3)}}$$

The solution for $l=3$ gives:

$$\boxed{A_3 = \frac{-7V(b^4+a^4)}{16(b^7-a^7)} \;;\; B_3 = \frac{7Va^4b^4(b^3+a^3)}{16(b^7-a^7)}}$$

Then:

$$\boxed{\begin{aligned}\Phi(r,\theta) \underset{l=0,1,2,3,4}{\approx} \cancel{V} \Bigg\{ & \underbrace{\frac{V}{2}}_{l=0} + \underbrace{\left\{\left[\frac{-3V(b^2+a^2)}{4(b^3-a^3)}\right] r + \frac{1}{r^2}\left[\frac{+3Va^2b^2(b+a)}{4(b^3-a^3)}\right]\right\} \underbrace{\cos\theta}_{P_1(\cos\theta)}}_{l=1} + \\ & + \left\{\left[\frac{-7V(b^4+a^4)}{16(b^7-a^7)}\right] r^3 + \frac{1}{r^4}\cdot\frac{7Va^4b^4(b^3+a^3)}{16(b^7-a^7)}\right\} \frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right) \Bigg\}\end{aligned}}$$

Let us now check some limits:

$b \to \infty$

$$\Phi(r,\theta) = V\left\{\frac{1}{2} + \frac{3a^2}{4r^2}\cos\theta + \frac{7a^4}{16r^4}\cdot\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right\}$$

We should compare this with (2.27), but (2.27) was done for

To properly compare first add a potential of a sphere all at V , which is $\frac{Va}{r}$, and then rescale everything so $V \to V/2$

$$\Phi = \frac{V}{2}\frac{a}{r} + \frac{3Va^2}{4r^2}\left[\cos\theta - \frac{7a^2}{12r^2}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right)\right]$$

(2.27)

adding Va/r and $V \to V/2$

which is almost what we want. Note that we cannot expect a perfect ~~result~~ agreement since the potential at $r \to \infty$ is still V for the lower hemisphere, while the θ-constant term in (2.27) above $\xrightarrow[r\to\infty]{} 0$.

Now do $a \to 0$ and a procedure similar to the one above works to compare with (2.27)