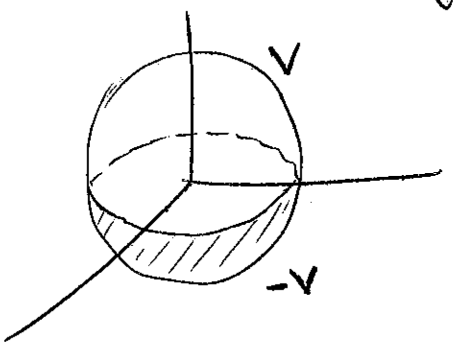
Problems related with Sec. 3.3

Show that (3.36) is correct

In class, we arrived to Eq. (3.35). Let us use it, as done in Sec. 3.3.

Consider $V(\theta) = \begin{cases} V & 0 \le \theta < \pi/2 \\ -V & \pi/2 < \theta \le \pi \end{cases}$



Then, $$A_l = \frac{2l+1}{2a^l} V \Bigg[\underbrace{\int_0^{\pi/2} P_l(\cos\theta)\sin\theta\, d\theta}_{\int_0^1 P_l(x)dx} - \underbrace{\int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta\, d\theta}_{\int_{-1}^0 P_l(x)dx} \Bigg]$$

$x = \cos\theta$

If $P_l(x)$ is even (under $x \to -x$), which occurs for l even,

then $\int_0^1 P_l(x)dx = \int_{-1}^0 P_l(x)\,dx$ and [...] above cancels.

If $P_l(x)$ is odd under $x \to -x$, which is the case for l odd,

then $\int_0^1 P_l(x)\,dx = -\int_{-1}^0 P_l(x)\,dx$

Then:

$$A_\ell = \frac{(2\ell+1)V}{a^\ell} \underbrace{\int_0^1 P_\ell(x)\,dx}$$

Using (3.15)

For $\ell = 1$

$$\int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = 1/2$$

For $\ell = 3$

$$\int_0^1 \frac{1}{2}(5x^3 - 3x)\,dx = \frac{5}{2}\left.\frac{x^4}{4}\right|_0^1 - \frac{3}{2}\left.\frac{x^2}{2}\right|_0^1 =$$

$$= \frac{5}{8} - \frac{3}{4} = -\frac{1}{8}$$

etc, etc.

$$A_1 = \frac{3V}{a}\,1/2 \quad ; \quad A_3 = \frac{7V}{a^3}\left(-\frac{1}{8}\right)$$

Putting all together:

$$\Phi(r,\theta) \underset{\substack{\uparrow \\ (3.33) \\ \text{with } B_\ell \equiv 0 \\ \text{since } r<a}}{=} A_1\, r\, P_1(\cos\theta) + \cdots$$

$$= \frac{3V}{a}\frac{1}{2}\, r\, P_1(\cos\theta) - \frac{7V}{8a^3}\, r^3 P_3(\cos\theta) + \cdots$$

$$= \boxed{V\left[\frac{3r}{2a} P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \cdots\right]}$$

(3.36) We leave $\ell = 5$ to the reader!