Problem 4.7, part (a) only

$$\rho(\vec{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta$$

Note that the problem has azimuthal symmetry. Then, $m=0$ and I can use Legendre polynomials. Moreover, $\sin^2\theta$ is a very simple function and I can "by eye" write it as a linear combination of $P_l(\cos\theta)$'s simply looking at Eq. (3.15):

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left[\frac{2P_2(\cos\theta)+1}{3}\right] =$$

$$= \frac{2}{3}\Big(\underbrace{1}_{P_0(\cos\theta)} - P_2(\cos\theta)\Big) = \frac{2}{3}\left[P_0(\cos\theta) - P_2(\cos\theta)\right]$$

$$q_{lm} \underset{(4.3)}{=} \int Y_{lm}^{*}(\theta',\phi')\, r'^{l} \rho(\vec{x}')\, d^3x' \underset{\substack{m=0 \\ \text{and use our } \rho(\vec{r})}}{=}$$

$$= \int Y_{l0}^{*}(\theta',\phi')\, r'^{l} \frac{1}{64\pi} r'^{2} e^{-r'} \frac{2}{3}\left[P_0(\cos\theta') - P_2(\cos\theta')\right].$$

$$\cdot\, r'^2 dr'\, d\phi' \sin\theta' d\theta'$$

Use now (3.57):

$$Y_{\ell 0}(\theta', \phi') = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta')$$

Then:

$$q_{\ell 0} \underset{\uparrow}{=} \underbrace{\frac{2\pi}{64\pi}\,\frac{2}{3}}_{1/48} \sqrt{\frac{2\ell+1}{4\pi}} \underbrace{\int_0^\infty r'^2\, r'^2\, r'^\ell\, e^{-r'}\, dr'}_{\underbrace{\int_0^\infty r'^{\ell+4} e^{-r'} dr'}_{\text{This is a } \cancel{\text{Bessel}} \text{ gamma function } \Gamma(\ell+5)\ *}} \underbrace{\int_0^\pi d\theta' \sin\theta'\, P_\ell(\cos\theta') \left[P_0(\cos\theta') - P_2(\cos\theta')\right]}_{\text{Use (3.21)} \quad \int_{-1}^{1} dx\, P_{\ell'}(x) P_\ell(x) = \frac{2}{2\ell+1}\delta_{\ell'\ell}}$$

$$\int_0^{2\pi} d\phi' = 2\pi$$

$$= \frac{1}{48}\sqrt{\frac{2\ell+1}{4\pi}}\, \Gamma(\ell+5) \left(2\delta_{\ell,0} - \frac{2}{5}\delta_{\ell,2}\right)$$

Then,

$$q_{00} = \frac{1}{48}\sqrt{\frac{1}{4\pi}}\, \Gamma(5)\, 2 = \boxed{\sqrt{\frac{1}{4\pi}}}$$

$$q_{20} = \frac{1}{48}\sqrt{\frac{5}{4\pi}}\, \Gamma(7)\left(-\frac{2}{5}\right) = \boxed{-6\sqrt{\frac{5}{4\pi}}}$$

$$*\ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

From Wikipedia:

$$\Gamma(z+5) = \int_0^\infty t^{z+4} e^{-t}\, dt$$

$$\Gamma(5) = 4! = 4.3.2.1 = 24$$

$$\Gamma(7) = 6! = 6.5.4.3.2.1 = 30.24$$

Now we need the potential at large distance:

$$\Phi = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} q_{l0} \frac{Y_{l0}(\theta,\phi)}{r^{l+1}} \qquad Y_{l0}(\theta,\phi) \xrightarrow{(3.57)} \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)$$

m=0 in this case

$$= \frac{1}{4\pi\epsilon_0}\left[4\pi \underbrace{q_{00}}_{\sqrt{1/4\pi}} \sqrt{\frac{1}{4\pi}}\, \underbrace{P_0(\cos\theta)}_{1} \frac{1}{r} + \frac{4\pi}{5} \underbrace{q_{20}}_{-6\sqrt{\frac{5}{4\pi}}} \sqrt{\frac{5}{4\pi}} \frac{P_2(\cos\theta)}{r^3} \right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{6}{r^3} P_2(\cos\theta) \right]$$

(3.15) says $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$

$$= \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{6}{r^3} \frac{1}{2}(3\cos^2\theta - 1) \right]}$$

This charge distribution has both a monopole and dipole moment.