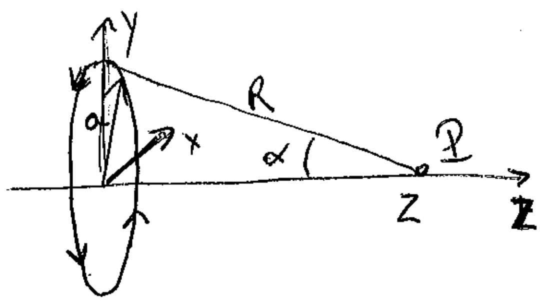
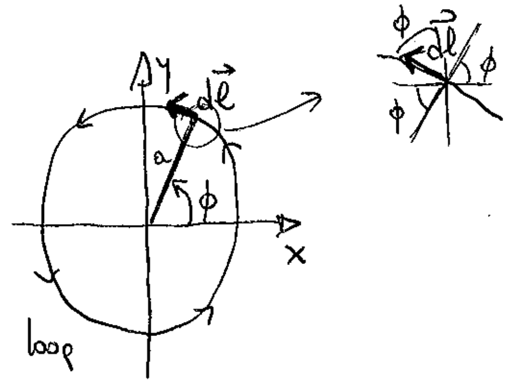
# Problem 5.3

We will solve the problem using the Biot-Savart law discussed in Section 5.2, starting with the magnetic field created by just a single loop of wire:



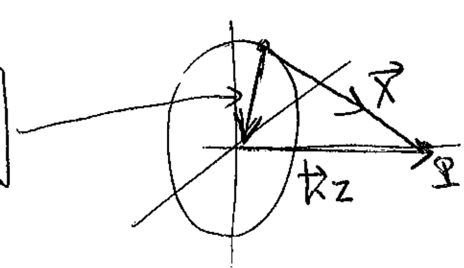




By symmetry only the z component matters at the axis.

$$B_z = \frac{\mu_0}{4\pi} I \int_{\text{loop}} \frac{(\vec{dl} \times \vec{X})|_z}{|\vec{X}|^3}$$

$\vec{X}$ = vector from the element of length $\vec{dl}$ to the observation point P.

$$\vec{X} = -\vec{i}\, a\cos\phi \\ -\vec{j}\, a\sin\phi \\ \vec{k}\, z$$



Now the element "$d\ell$"

$$d\vec{\ell} = -\vec{\imath}\,\overbrace{a\,d\phi}^{|d\vec{\ell}|}\sin\phi + \vec{\jmath}\,a\,d\phi\cos\phi$$

Then:
$$d\vec{\ell}\times\vec{X} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -a\,d\phi\sin\phi & a\,d\phi\cos\phi & 0 \\ -a\cos\phi & -a\sin\phi & z \end{vmatrix}$$

$$d\vec{\ell}\times\vec{X}\Big|_z = a^2\,d\phi\sin^2\phi + a^2\,d\phi\cos^2\phi = a^2\,d\phi$$

$$B_z = \frac{\mu_0}{4\pi} I \frac{1}{R^3}\int_0^{2\pi} a^2\,d\phi = \boxed{\frac{\mu_0 I a^2}{2R^3}} = \underbrace{\frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}}}$$

$|\vec{X}| = R$

Note that Eq. (5.40) of Jackson should reduce to this if $\theta = 0$ and $r$ becomes $z$ and indeed it does.

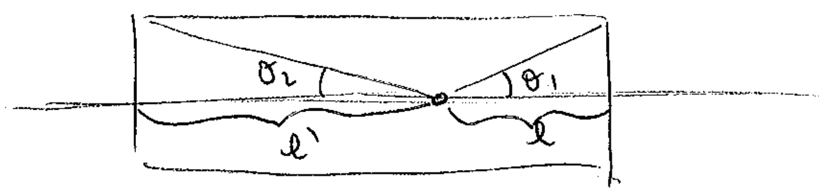
Then, we must integrate over all rings. For this purpose we must consider that the "$B_z$" we calculated is just a "small" part of the total and integrate.

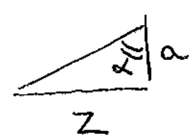
$$dB_z = \frac{\mu_0 a^2 \text{"}dI\text{"}}{2(a^2+z^2)^{3/2}} \longrightarrow I\,N\,dz$$

$N$: turns per unit length

$N\,dz$ = turns in interval $dz$

$$\overset{\text{total}}{B_z} = \frac{\mu_0 I N a^2}{2} \int_{-\ell'}^{\ell} \frac{dz}{(a^2+z^2)^{3/2}}$$




Introduce $z = a\tan\alpha$

$$\sqrt{a^2+z^2} = a\sqrt{1+\left(\frac{z}{a}\right)^2} = a\sqrt{1+\tan^2\alpha} = a\sqrt{1+\frac{\sin^2\alpha}{\cos^2\alpha}} =$$

$$= \frac{a}{\cos\alpha}$$

$$dz = a\,d(\tan\alpha) = a\,d\left(\frac{\sin\alpha}{\cos\alpha}\right) = a\,\frac{d(\sin\alpha)}{\cos\alpha} + \sin\alpha\,d\left(\frac{1}{\cos\alpha}\right) =$$

$$= a\left[\frac{\cos\alpha}{\cos\alpha}\,d\alpha + \sin\alpha\left(\frac{-1}{\cos^2\alpha}\right)(-\sin\alpha)\,d\alpha\right] = \frac{a}{\cos^2\alpha}\,d\alpha$$

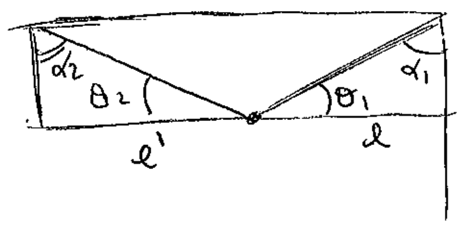
$$\frac{dz}{(a^2+z^2)^{3/2}} = \frac{\frac{a}{\cos^2\alpha}\,d\alpha}{\left(\frac{a}{\cos\alpha}\right)^3} = \frac{1}{a^2}\cos\alpha\,d\alpha$$

$$\ell = a \tan\alpha_1$$

$$\ell' = a \tan\alpha_2$$

$$\int_{-\ell'}^{\ell} \frac{dz}{(a^2+z^2)^{3/2}} = \int_{-a\tan\alpha_2}^{a\tan\alpha_1} \frac{dz}{(a^2+z^2)^3} = \int_{-\alpha_2}^{\alpha_1} \frac{\cos\alpha \, d\alpha}{a^2} = \frac{1}{a^2} \sin\alpha \Big|_{\alpha_2}^{\alpha_1}.$$

$$B_z^{total} = \frac{\mu_0 I N}{2} (\sin\alpha_1 + \sin\alpha_2) = \boxed{\frac{\mu_0 I N}{2}(\cos\theta_1 + \cos\theta_2)}$$




$$\sin\alpha_1 = \frac{\ell}{\sqrt{\ell^2+a^2}}$$

$$\cos\theta_1 = \frac{\ell}{\sqrt{\ell^2+a^2}}$$