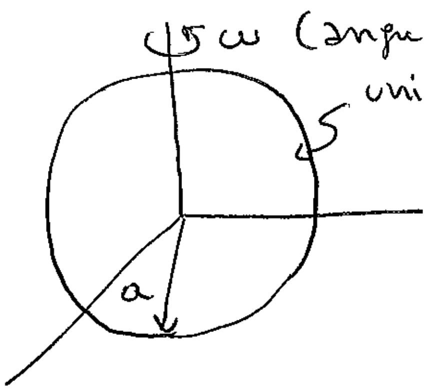
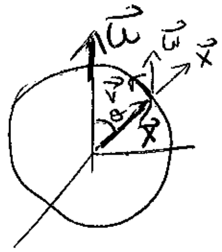
Problem 5.13



Find $\vec{A}$ and $\vec{B}$.

The velocity $\vec{v}$ at any point on the sphere is



$$\vec{v} = \vec{\omega} \times \vec{x} = \omega \times (\hat{\omega} \times \hat{x}) = \omega |\vec{x}| \hat{\phi} \sin\theta$$

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

Then, the surface current density is $\vec{K} = \sigma \vec{v} =$

$= \sigma \omega a \sin\theta \, \hat{\phi}$

using notation of (5.87)

The charge exists <u>only</u> at the surface.
Then, outside the surface $\vec{J} = 0$ and $\nabla \times \vec{H} = 0$
and $\vec{H} = -\nabla \Phi_M$.

For $r < a$ we propose, following (3.33)

$$\Phi_M(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

For $r > a$, we also use (3.33)

$$\Phi_M(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

At $r=a$, there are two boundary conditions (see (5.86) and (5.87)):

$$(\vec{B}_2-\vec{B}_1)\cdot\vec{n}_{21}=0$$

$$\vec{n}_{21}\times(\vec{H}_2-\vec{H}_1)=\vec{K}$$

The first one says that the "r" component of B is continuous:

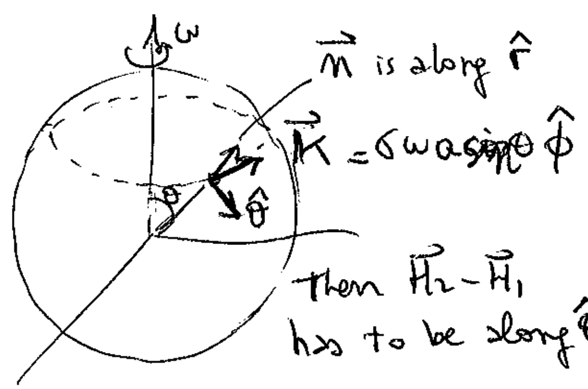
$$-\mu_0 \left.\frac{\partial \Phi_M^{r<a}}{\partial r}\right|_a = -\mu_0 \left.\frac{\partial \Phi_M^{r>a}}{\partial r}\right|_a$$

for $\vec{B}=\mu_0\vec{H}=\mu_0(-\nabla\Phi_M)$ but μ_0 drops.

$$\sum_{l=0}^{\infty} A_l\, l\, a^{l-1} P_l(\cos\theta) = \sum_{l=0}^{\infty} B_l \frac{(-l-1)}{a^{l+2}} P_l(\cos\theta)$$

or

$$\boxed{A_l\, l\, a^{l-1} = -(l+1)\frac{B_l}{a^{l+2}}}$$



The other equation says:

$$-\left[\left.\frac{1}{r}\frac{\partial \Phi_M^{r>a}}{\partial \theta}\right|_a - \left.\frac{1}{r}\frac{\partial \Phi_M^{r<a}}{\partial \theta}\right|_a\right] = \sigma\omega a\sin\theta$$

$$-\left[\frac{1}{a}\sum_{l=0}^{\infty}\frac{B_l}{a^{l+1}}\frac{\partial}{\partial\theta}P_l(\cos\theta) - \frac{1}{a}\sum_{l=0}^{\infty}A_l a^l \frac{\partial}{\partial\theta}P_l(\cos\theta)\right] = \sigma\omega a \underbrace{\sin\theta}_{-\frac{\partial P_1}{\partial\theta}(\cos\theta)}$$

since $P_1(\cos\theta) = \cos\theta$

Then:

$$\boxed{-\frac{1}{a}\frac{B_l}{a^{l+1}} + \frac{1}{a}A_l a^l = -\sigma\omega a\,\delta_{l1}}$$

Consider $l \neq 1$ first. Then:

$$A_l\, l\, a^{l-1} = -(l+1)\frac{B_l}{a^{l+2}} \quad \text{or} \quad A_l a^{l-1} = -\frac{(l+1)}{l}\frac{B_l}{a^{l+2}}$$

$$\frac{A_l a^l}{a} = \frac{B_l}{a^{l+2}} \quad \text{or} \quad A_l a^{l-1} = \frac{B_l}{a^{l+2}} \rightarrow \text{incompatible}$$

The only solution is $\boxed{A_l = B_l = 0 \text{ for } l \neq 1}$.

Consider now $l = 1$:

$$A_1 = -\frac{2B_1}{a^3}$$

$$-\frac{B_1}{a^3} + A_1 = -\sigma\omega a$$

or $A_1 = +2(-\sigma\omega a - A_1) \rightarrow 3A_1 = -2\sigma\omega a$

$$\boxed{A_1 = \frac{-2\sigma\omega a}{3}}$$

$$B_1 = -\frac{a^3}{2} A_1 = -\frac{a^3}{2}\left(-\frac{2\sigma\omega a}{3}\right) = \boxed{\frac{1}{3}\sigma\omega a^4}$$

Then:

$$\Phi^{r<a}(r,\theta) = -\frac{2\sigma\omega a}{3}\, r\cos\theta$$

$$\Phi^{r>a}(r,\theta) = \frac{1}{3}\sigma\omega a^4\, \frac{1}{r^2}\cos\theta$$

We can get $\vec{H}$ via $\vec{H} = -\nabla\Phi_M = \mu_0^{-1}\vec{B}$

$$\vec{B} = \mu_0(-\nabla\Phi_M) = \mu_0\left[-\frac{\partial\Phi_M}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\Phi_M}{\partial\theta}\hat{\theta}\right] =$$

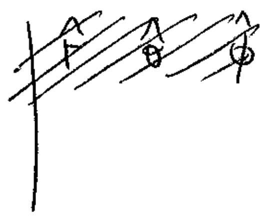
back cover of book ; no ϕ dependence.

$$= \begin{cases} -\mu_0\left[\left(-\frac{2\sigma\omega a}{3}\right)\cos\theta\,\hat{r} + \frac{1}{r}\left(\frac{2}{3}\sigma\omega a\right) r(-\sin\theta)\,\hat{\theta}\right] & r<a \\ -\mu_0\left[\left(\frac{1}{3}\sigma\omega a^4\right)\left(\frac{-2}{r^3}\right)\cos\theta\,\hat{r} + \frac{1}{r}\left(\frac{1}{3}\sigma\omega a^4\right)\frac{1}{r^2}(-\sin\theta)\,\hat{\theta}\right] & r>a \end{cases}$$

$$= \boxed{\begin{cases} \frac{2\sigma\omega a\mu_0}{3}\underbrace{\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)}_{\hat{k}} & r<a \\ \frac{2}{3}\sigma\omega a^4\mu_0\left(\frac{1}{r^3}\cos\theta\,\hat{r} + \frac{1}{2}\frac{\sin\theta}{r^3}\,\hat{\theta}\right) & r>a \end{cases}}$$

Now we have to find $\vec{A}$. We can use $\nabla \times \vec{A} = \vec{B}$.
We know that $\vec{A}$ is along $\hat{\phi}$, by symmetry.

Then:



From back cover of book, we know $\nabla \times \vec{A}$ in spherical coordinates. Use $A_1 = A_2 = 0$ (which means $A_r = A_\theta = 0$) and you arrive to:

$$\nabla \times \vec{A} = \nabla \times (A_\phi \hat{\phi}) = \hat{e}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\phi) - \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)\,\hat{e}_\theta$$

Then, for $r < a$ we have to find a A_ϕ such that

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\phi) = \frac{2\sigma\omega a\,\mu_0 \cos\theta}{3}$$

and

$$-\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi) = -\frac{2\sigma\omega a}{3}\mu_0 \sin\theta$$

By inspection $\boxed{A_\phi = +\frac{\mu_0\sigma\omega a}{3} r\sin\theta}$ $(r < a)$

$$\text{check}\begin{cases} -\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi) = -\frac{1}{r}\frac{\partial}{\partial r}(r^2\sin\theta)\left(+\frac{\mu_0\sigma\omega a}{3}\right) = +\frac{2\cancel{r}\sin\theta}{\cancel{r}}\,\frac{\mu_0\sigma\omega a}{3} = B_{\theta\,(r<a)} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(+\frac{\mu_0\sigma\omega a}{3}\right) r\sin\theta\right) = +\frac{\mu_0\sigma\omega a}{3}\,\frac{1}{\sin\theta}\underbrace{\frac{\partial}{\partial\theta}\sin^2\theta}_{2\sin\theta\cos\theta} = \frac{2}{3}\mu_0\sigma\omega a\cos\theta = B_{r\,(r<a)} \end{cases}$$

For $r > a$, $\boxed{A_\phi = \frac{\mu_0 \sigma \omega a^4}{3r^2} \sin\theta}$

Check:

$$-\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi) = -\frac{1}{r}\,\frac{\mu_0\sigma\omega a^4}{3}\sin\theta \underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\right)}_{-\frac{1}{r^2}} = \underbrace{+\frac{\mu_0\sigma\omega a^4 \sin\theta}{3r^3}}_{B_\theta \text{ for } r>a} \checkmark$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\mu_0\sigma\omega a^4}{3r^2}\sin\theta\right) =$$

$$= \frac{\mu_0\sigma\omega a^4}{3r^3}\,\frac{1}{\sin\theta}\underbrace{\frac{\partial}{\partial\theta}\sin^2\theta}_{2\sin\theta\cos\theta} = \underbrace{\frac{2}{3}\frac{\mu_0\sigma\omega a^4\cos\theta}{r^3}}_{B_r \text{ for } r>a} \checkmark$$