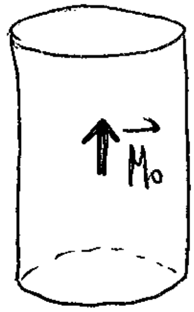
Problem 5.19



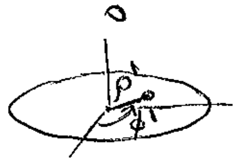
Find $\vec{H}$ and $\vec{B}$ on the axis.

Since there are no currents, then $\nabla \times \vec{H} = 0$ or $\vec{H} = -\nabla \Phi_M$.

From (5.99) we know $\sigma_M = \vec{n}\cdot\vec{M}$. This will be nonzero only at the top and bottom of the cylinder. Since $\nabla \cdot M = 0$ inside and outside, then we can use the second term in (5.100) to get Φ_M.

$$\Phi_M(\vec{x}) = \frac{1}{4\pi}\oint_S \frac{\vec{n}'\cdot\vec{M}(\vec{x}')\,da'}{|\vec{x}-\vec{x}'|} = \frac{1}{4\pi}M_0\int_{top}\frac{da'}{|\vec{x}-\vec{x}'|} - \frac{M_0}{4\pi}\int_{bottom}\frac{da'}{|\vec{x}-\vec{x}'|}$$

$$\int da' = \int_0^a \rho'\,d\rho' \int_0^{2\pi} d\phi'$$



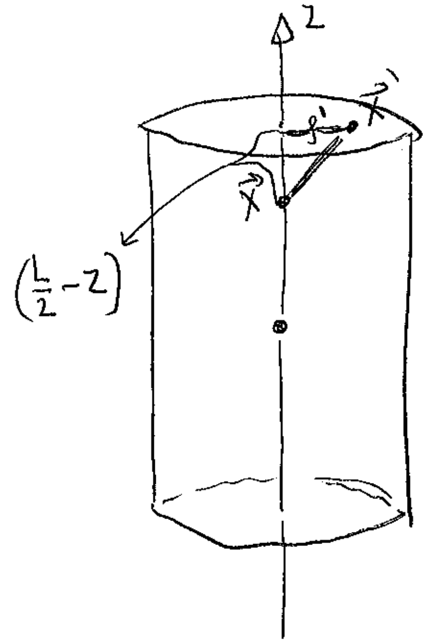
Since $\vec{x}$ is on the axis, then $\vec{x} = (0, 0, z)$.

$$\vec{x}' = (\rho', \phi', \pm b/2)$$

top and bottom

cylindrical coordinates require ρ, ϕ, and z

Then, $|\vec{x}-\vec{x}'| = \begin{cases} \sqrt{\rho'^2 + \left(z-\frac{L}{2}\right)^2} & \text{top} \\ \sqrt{\rho'^2 + \left(z+\frac{L}{2}\right)^2} & \text{bottom} \end{cases}$



Thus:

$$\Phi_M = \frac{M_0}{4\pi}\left[\int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi'}{\sqrt{\rho'^2+(z-L/2)^2}} - \int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi'}{\sqrt{\rho'^2+(z+L/2)^2}}\right]$$

$$= \frac{M_0}{2}\left(\sqrt{\rho'^2+(z-L/2)^2}\,\Big|_0^a - \sqrt{\rho'^2+(z+L/2)^2}\,\Big|_0^a\right)$$

Consider $z < L/2$, In this case:

$$\Phi_M = \frac{M_0}{2}\left[\sqrt{a^2+\left(z-\frac{L}{2}\right)^2} - \sqrt{a^2+\left(z+\frac{L}{2}\right)^2} - \underbrace{\left(\sqrt{\left(z-\frac{L}{2}\right)^2} - \sqrt{\left(z+\frac{L}{2}\right)^2}\right)}_{\text{lower limit "0"}}\right]$$

upper limits "a"

If $z < L/2$, then the last $\sqrt{}$ becomes

$$\sqrt{\left(\frac{L}{2}-z\right)^2} - \sqrt{(z+L/2)^2} = \left(\frac{L}{2}-z\right) - (z+L/2) = -2z$$

$$\boxed{\Phi_M = \frac{M_0}{2}\left[\sqrt{a^2+\left(\frac{L}{2}-z\right)^2} - \sqrt{a^2+\left(\frac{L}{2}+z\right)^2} + 2z\right]}$$

For $z > L/2$, the last $\sqrt{\ }$ becomes

$$\sqrt{(z-L/2)^2} - \sqrt{(z+L/2)^2} = z - L/2 - (z+L/2) = -L$$

and $$\boxed{\Phi_M = \frac{M_0}{2}\left[\sqrt{a^2 + (z-L/2)^2} - \sqrt{a^2 + (z+L/2)^2} + L\right]}$$

Since $\vec{H} = -\nabla \Phi_M$, $H_z = -\frac{\partial}{\partial z}\Phi_M$ and

$z < L/2$

$$H_z = -\frac{M_0}{2}\left[\frac{-\left(\frac{L}{2}-z\right)}{\sqrt{a^2+\left(\frac{L}{2}-z\right)^2}} - \frac{\left(\frac{L}{2}+z\right)}{\sqrt{a^2+\left(\frac{L}{2}+z\right)^2}} + 2\right]$$

$z > L/2$

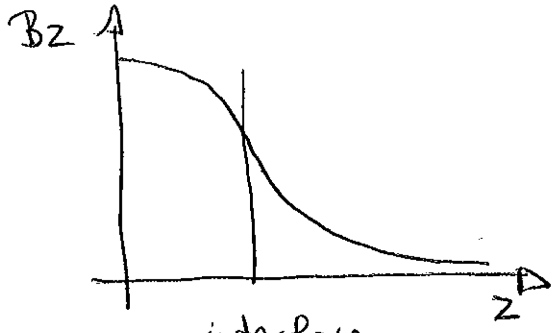
$$H_z = -\frac{M_0}{2}\left[\frac{(z-L/2)}{\sqrt{a^2+(z-L/2)^2}} - \frac{(z+L/2)}{\sqrt{a^2+(z+L/2)^2}}\right]$$

Using $\vec{B} = \mu_0(\vec{H}+\vec{M})$ we can easily deduce $\vec{B}$:
For $z > L/2$, there is no $\vec{M}$. Thus, $B_{z>L/2} = \mu_0 H_{z>L/2}$

$$\boxed{B_z = -\frac{\mu_0 M_0}{2}\left[\frac{(z-L/2)}{\sqrt{a^2+(z-L/2)^2}} - \frac{(z+L/2)}{\sqrt{a^2+(z+L/2)^2}}\right]}$$

For $z < L/2$, H_z differs from $H_{z\,z>L/2}$ in the constant $-\frac{M_0}{2}\cdot 2 = -M_0$. But that is precisely what will be cancelled by $\vec{M}$ in $(\vec{H}+\vec{M})$. Thus $\boxed{B_{z\,z<L/2} \equiv B_{z\,z>L/2}}$

Then B_z is continuous across the interface and slowly decays with z outside the cylinder (to see this you have to expand the Γ's in the expression for B_z, at large z)



H_z is <u>not</u> continuous. Actually

$$\lim_{z \to L/2} H_z^{z<L/2} = -\frac{M_0}{2}\left(\frac{-L}{\sqrt{a^2+L^2}} + 2\right)$$

$$= \frac{M_0}{2} \cdot \frac{L}{\sqrt{a^2+L^2}} - M_0 \quad (< 0 \text{ by the way})$$

$$\lim_{z \to L/2} H_z^{z>L/2} = -\frac{M_0}{2}\left(\frac{-L}{\sqrt{a^2+L^2}}\right) = \frac{M_0}{2} \cdot \frac{L}{\sqrt{a^2+L^2}}$$

Then $\Delta H_z = \lim_{z \to L/2}\left(H_z^{z>L/2} - H_z^{z<L/2}\right) = M_0$