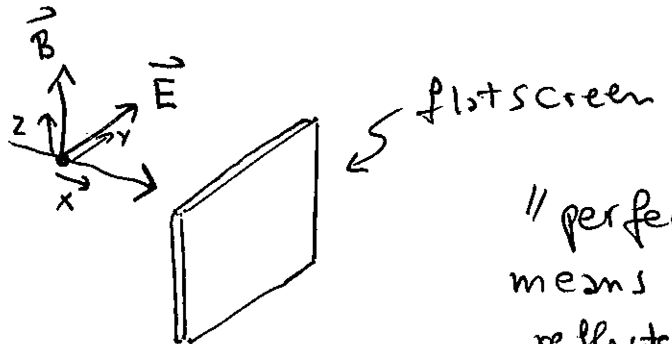
Problem 11 of Chapter 6

We will assume that the $\vec{E}$ and $\vec{B}$ fields point along the y and z directions, respectively, as shown in the sketch:



"perfectly absorbing" means there is no reflected wave to be considered.

The formula for the Maxwell stress tensor is:

$$T_{\alpha\beta} \underset{\underset{(6.120)}{\uparrow}}{=} \epsilon_0 \left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\left(\vec{E}.\vec{E} + c^2 \vec{B}.\vec{B}\right)\delta_{\alpha\beta} \right]$$

Since $E_2 \neq 0$, but $E_3 = E_1 = 0$ and $B_3 \neq 0$, but $B_1 = B_2 = 0$ then the nondiagonal terms are always 0: $T_{\alpha\beta} = 0$ if $\alpha \neq \beta$.

For the diagonal terms:

$$T_{11} = \epsilon_0 \left[\underset{=0}{E_1^2} + c^2 \underset{=0}{B_1^2} - \frac{1}{2}\left(\vec{E}.\vec{E} + c^2\vec{B}.\vec{B}\right)\right] = -\frac{\epsilon_0}{2}\left(E^2 + c^2 B^2\right)$$

$$T_{22} = \epsilon_0 \left[\underset{\neq 0}{E_2^2} + c^2 \underset{=0}{B_2^2} - \frac{1}{2}\left(\vec{E}.\vec{E} + c^2\vec{B}.\vec{B}\right)\right] \underset{\uparrow}{=} \epsilon_0\left(E^2 - \frac{1}{2}\left(E^2 + c^2B^2\right)\right)$$

$E_2 = E$ since $\vec{E}$ points only along one axis

$$= \epsilon_0 \left(\frac{E^2 - c^2B^2}{2}\right)$$

$$T_{33} = \epsilon_0 \Big[\underbrace{E_3^2}_{=0} + c^2 \underbrace{B_3^2}_{=B^2} - \frac{1}{2}(\vec{E}\cdot\vec{E} + c^2 \vec{B}\cdot\vec{B})\Big]$$

$$= \epsilon_0 \left(c^2 B^2 - \frac{1}{2}(E^2 + c^2 B^2)\right) = \epsilon_0 \left(\frac{B^2 c^2}{2} - \frac{E^2}{2}\right)$$

Note that $\vec{E} = E_0 \cos(\cancel{k}x - \omega t)\,\hat{j}$

$\vec{B} = \frac{E_0}{c} \cos(kx - \omega t)\,\hat{k}$

Then: $T_{11} = -\frac{\epsilon_0}{2}\left(E_0^2 \cos^2(kx - \omega t) + c^2 \frac{E_0^2}{c^2} \cos^2(kx - \omega t)\right)$

$$= -\epsilon_0 E_0^2 \cos^2(kx - \omega t)$$

$$T_{22} = T_{33} = 0$$

The actual formula to be used is (6.122):

$$\frac{d}{dt}\left(\vec{P}_{mech} + \vec{P}_{field}\right)_\alpha = \oint_S \sum_\beta T_{\alpha\beta}\, n_\beta \, da$$

For the pressure on the screen we only need the x component, $\alpha = 1$:

$$\sum_\beta T_{\alpha\beta}\, n_\beta = \sum_\beta T_{1\beta}\, n_\beta = T_{11} \underbrace{n_1}_{} = \pm T_{11}$$

± 1 depending on whether we are along or opposite to x axis.

and $\oint_S \sum_\beta T_{\alpha\beta}\, n_\beta \, da = \pm T_{11} \cdot \text{Area}$

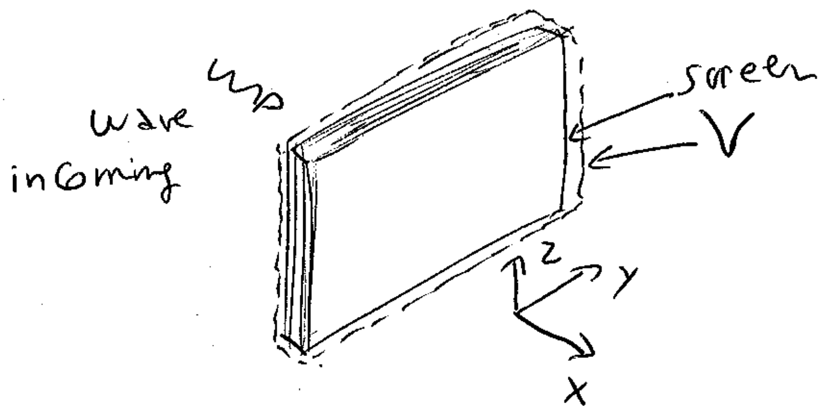
$$= \mp \epsilon_0 E_0^2 \cos^2(kx - \omega t) \cdot \text{Area}$$

$x=0$ if that is the location of the wall.

Area along y-z plane

To clarify well the signs, we need to think carefully about the volume used:

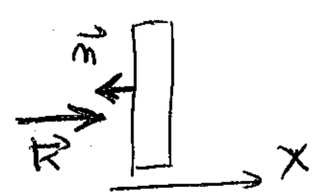
We will simply take a volume V that corresponds to the volume of the screen.



Only one side is important obviously. Also the term $\frac{d}{dt}\vec{P}_{field}$ inside the volume is 0, since V has no empty space. The term $\frac{d}{dt}\vec{P}_{mechanical}$ will be the force caused by the radiation when it is absorbed by the screen. Thus:

$$\underset{\text{on Screen}}{\overrightarrow{\text{Force}}} = \oint_S \sum_\beta T_{1\beta}\, n_\beta \, da = -T_{11} \cdot \text{Area screen.}$$

because $n_1 = -1$



But $T_{11} = -\epsilon_0 E_0^2 \cos^2(kx-\omega t)$ and it is negative, thus the overall force is positive, i.e. the screen is pushed into the positive x direction, as it has to be.

$$\frac{|\vec{\text{Force}}|}{\text{Area}_{\text{screen}}} = \text{Pressure} \underset{\uparrow}{=} \epsilon_0 E_0^2 \cos^2(\omega t)$$

if we say the surface that absorbs the plane wave is at $x=0$.

We can average over time using

$$\frac{1}{T}\int_0^{T=2\pi/\omega} dt \cos^2(\omega t) = \frac{1}{T}\int_0^T dt \cdot \frac{1}{2}\left(1+\cos(2\omega t)\right)$$

$$= \frac{1}{2} + 0 = \frac{1}{2}.$$

$$\boxed{\langle \text{pressure} \rangle_{\substack{\text{time}\\ \text{averaged}}} = \frac{\epsilon_0 E_0^2}{2}}$$

radiation pressure

Let us calculate the energy of the plane wave and compare. The energy per unit volume is:

$$u = \frac{\epsilon_0}{2}\left(\vec{E}^2 + c^2\vec{B}^2\right)$$

But note that the T_{11} previously calculated is precisely this number, up to a sign, i.e.

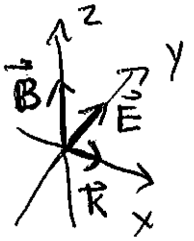
$$u = -T_{11}$$

Since pressure was shown to be $\dfrac{\overset{\text{due to } n_1}{-T_{11}} \cdot \text{Area } \cancel{\text{screen}}}{\text{Area } \cancel{\text{screen}}}$

Then: $\boxed{\text{pressure} = u}$

It is also interesting to calculate the momentum of the wave. For this we need

$$\vec{P}_{field} = \epsilon_0 \int_V \underbrace{(\vec{E} \times \vec{B})}_{|\vec{E}||\vec{B}|\,\hat{e}_x} \, d^3x =$$



$$= \epsilon_0 \frac{E_0^2}{c} \int \cos^2(kx - \omega t)\, \hat{e}_x \, d^3x$$

drop integral

The momentum density averaged over time is :

gives $1/2$

$$\vec{P}_{field} = \frac{\epsilon_0 E_0^2}{c\, 2} \hat{e}_x$$

Thus

$$\boxed{\vec{P}_{field} = \frac{1}{c} u \,\hat{e}_x}$$

energy density of wave, time averaged

This makes sense for plane waves where energy = c · momentum ($E = cp$)

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