Griffiths, page 328, Example 16 (2nd edition)

From Homework problem (1.4) (b) we know that the electric field inside a uniformly charged sphere is:

$$|E \text{ inside}| = \frac{Qr}{4\pi\epsilon_0 R^3}$$, if R is the radius

and it points along $\hat{e}_r$. At the surface $r=R$:

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2}\hat{e}_r$$

where $\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

By symmetry the net force is along the z axis. Then, only we care about the z component of the Maxwell stress tensor:

$$\sum_\beta T_{\alpha\beta} n_\beta = \sum_\beta T_{z\beta} n_\beta = T_{zx} n_x + T_{zy} n_y + T_{zz} n_z$$

$$T_{zx} = \epsilon_0 E_z E_x = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \cos\theta(\sin\theta\cos\phi)$$

$$T_{zy} = \epsilon_0 E_z E_y = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \cos\theta(\sin\theta\sin\phi)$$

$$T_{zz} = \frac{\epsilon_0}{2}\left(E_z^2 - E_x^2 - E_y^2\right) = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \underbrace{\left(\cos^2\theta - \sin^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi\right)}_{\cos^2\theta - \sin^2\theta}$$

$$\epsilon_0\left(E_z^2 - \frac{1}{2}(E_z^2 + E_x^2 + E_y^2)\right)$$

$$n_x = \sin\theta\cos\phi$$
$$n_y = \sin\theta\sin\phi$$
$$n_z = \cos\theta$$

$$\sum_\beta T_{z\beta} n_\beta = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \underbrace{\Big[\overbrace{\cos\theta\sin^2\theta\cos^2\phi + \cos\theta\sin^2\theta\sin^2\phi}^{\cos\theta\sin^2\theta} + \frac{1}{2}\underbrace{(\cos^2\theta - \sin^2\theta)}_{1-2\sin^2\theta}\cos\theta \Big]}_{\cancel{\cos\theta\sin^2\theta} + \frac{\cos\theta}{2} - \cancel{\sin^2\theta\cos\theta}}$$

$$= \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \cos\theta .$$

$$da = R^2 \sin\theta \, d\theta \, d\phi$$

Then: $$\sum_\beta T_{z\beta} n_\beta \, da = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \cos\theta\sin\theta \, d\theta \, d\phi$$

Integrating over the upper surface:

$$\oint_S \sum_\beta T_{z\beta} n_\beta \, da = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi/2} \cos\theta\sin\theta \, d\theta}_{\frac{\sin^2\theta}{2}\Big|_0^{\pi/2} = 1/2}$$

$$= \frac{\cancel{\epsilon_0}}{2} \frac{Q^2}{\cancel{4\pi}\, 4\pi\, \epsilon_0^{\cancel{2}} R^2} \, \cancel{2\pi} \, \frac{1}{2}$$

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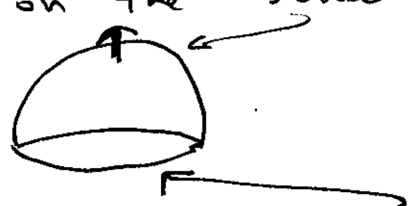
$$= \boxed{\frac{1}{4\pi\epsilon_0} \cdot \frac{Q^2}{8R^2}}$$

This is the force on the "bowl"

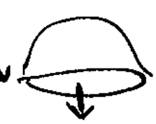


We still need
to calculate for the lower disk ($\theta = \pi/2$)

Here $n_x = n_y = 0$, $n_z = -1$

$$d\vec{a} = +(r\,d\phi)\,dr\,\hat{e}_z$$

The sign is already in n_β



From the values of n_β, we need only T_{zz}

$$\oint_{\substack{\text{lower}\\ \text{disk}}} T_{zz}\, n_z\, da = \frac{\epsilon_0}{2}\int_0^{2\pi}\!\!\int_0^{R} \left(E_z^2 - E_x^2 - E_y^2\right) r\,d\phi\,dr$$

(-1) from n_z

From Homework problem (1.4) (b)

$$|E_{inside}| = \frac{Q r}{4\pi\epsilon_0 R^3} \quad \text{pointing along } \hat{e}_r$$

Since $\theta = \pi/2$, $\hat{e}_r = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}$

Then, E_z in lower disk $= 0$ (reasonable by symmetry)

$$E_x = \frac{Q r}{4\pi\epsilon_0 R^3}\cos\phi$$

$$E_y = \frac{Q r}{4\pi\epsilon_0 R^3}\sin\phi$$

$$E_z^2 - E_x^2 - E_y^2 = \frac{Q^2 r^2}{(4\pi\epsilon_0 R^3)^2} \underbrace{\left(0 - \cos^2\phi - \sin^2\phi\right)}_{(-1)}$$

$$\oint_{\text{lower disk}} T_{zz}\, n_z\, da = \frac{\epsilon_0}{2}\int_0^{2\pi} d\phi \int_0^R r\,dr\ \frac{Q^2 r^2}{(4\pi\epsilon_0 R^3)^2}(-1)\underbrace{(-1)}_{n_z}$$

$$= \frac{\epsilon_0}{2}\cdot\frac{Q^2}{(4\pi\epsilon_0 R^3)^2}\cdot 2\pi \underbrace{\int_0^R r^3 dr}_{R^4/4} = \boxed{\frac{1}{4\pi\epsilon_0}\cdot\left(\frac{Q^2}{16 R^2}\right)}$$

This points along the z axis, in the positive direction (it's also repulsive).

$\frac{1}{4\pi\epsilon_0}\cdot\frac{Q^2}{8R^2}$

$\frac{1}{4\pi\epsilon_0}\cdot\frac{Q^2}{16R^2}$

$$\underline{\underline{\text{Total Force}}} = \oint_{\text{all surface}} \sum_\beta T_{z\beta} n_\beta\, da = \boxed{\frac{1}{4\pi\epsilon_0}\cdot\frac{3}{16}\frac{Q^2}{R^2}}.$$

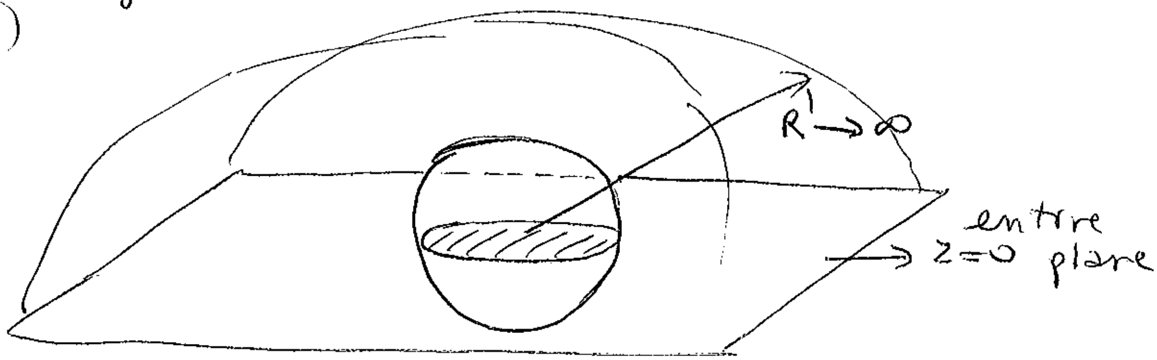
Force transmitted across the surface S acting on the particles and fields inside.

Note that $\frac{\partial}{\partial t}\vec{P}_{field} = 0$ since $\vec{E}$ is t-independent.

So this force is compensated by some mechanical device that keeps the charge in place, otherwise it would basically explode

See page 261, bottom, Jackson.

As Griffiths explains, we could use other geometries. For instance:



The contribution at $R' \to \infty$ is 0 because $\vec{E} \underset{R' \to \infty}{\Longrightarrow} 0$.

With regards to $z=0$, we already did the shaded circle. So now we have to handle the rest.

The electric field in that region is that of a point like particle at the origin, with charge Q,

$$|\vec{E}| = \frac{Q}{4\pi\epsilon_0 r^2} \quad \text{pointing away from the origin.}$$

$\hat{e}_r = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}$ as before ($\theta = \pi/2$)

$$E_x = \frac{Q}{4\pi\epsilon_0 r^2}\cos\phi \quad , \quad E_y = \frac{Q}{4\pi\epsilon_0 r^2}\sin\phi$$

We only need T_{zz} (by symmetry)

$$T_{zz} = \epsilon_0\left(E_z^2 - \frac{1}{2}(E_z^2 + E_x^2 + E_y^2)\right) \overset{E_z = 0}{=} -\frac{\epsilon_0}{2}(E_x^2 + E_y^2)$$

$$= -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 \underbrace{(\cos^2\phi + \sin^2\phi)}_{=1}$$

$da = r\,d\phi\,dr$ pointing in the $(-\hat{k})$ direction.

Then $\underset{z=0 \to S}{\oint} \sum_\beta T_{z\beta} n_\beta \, da =$

$$= \int_0^{2\pi} d\phi \int_R^\infty dr \cdot r \quad \frac{\epsilon_0}{2} \left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^4}$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \cancel{2\pi} \frac{\cancel{\epsilon_0}}{\cancel{2}} \frac{Q^2}{4\cancel{\pi}\cancel{\epsilon_0}} \underbrace{\int_R^\infty \frac{dr}{r^3}}_{\underbrace{\left. -\frac{1}{2r^2} \right|_R^\infty}_{\frac{1}{2R^2}}} = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q^2}{8R^2}$$

which is the same value as for the "bowl"