page 35 Jackson
Proof of
" $\frac{r^2 \nabla^2}{6}\rho$ "
term

Let us start with (1.30):

$$\nabla^2 \Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}\,')\,\frac{3a^2}{(r^2+a^2)^{5/2}}\,d^3x'$$

For simplicity in the algebra let us take $\vec{x}$ as the origin of coordinates $\vec{0}$. Then $r = |\vec{x}\,'|$.
For the expansion of ρ let us use "good old" cartesian coordinates as in page 150, Eq (4.22):

$$\rho(\vec{x}\,') \cong \rho(\vec{0}) + \vec{x}\,'\cdot\underbrace{\nabla\rho(\vec{0})}_{\text{indep. of } \vec{x}\,'} + \frac{1}{2}\sum_{ij} x'_i x'_j \underbrace{\left[\left.\frac{\partial^2\rho(\vec{0})}{\partial x'_i\,\partial x'_j}\right|_{\vec{x}\,'=\vec{0}}\right]}_{\text{independent of } \vec{x}\,'}$$

The linear term becomes

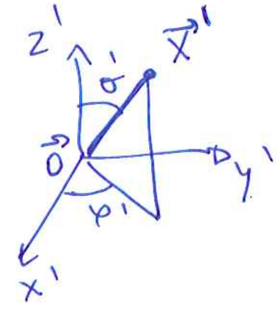
$$-\frac{1}{4\pi\epsilon_0}\int \vec{x}\,'\cdot\nabla\rho(\vec{0})\,\frac{3a^2}{(r^2+a^2)^{5/2}}\,d^3x'$$

$\vec{x}\,' = (x', y', z')$

$x' = r\sin\theta'\cos\varphi'$

$y' = r\sin\theta'\sin\varphi'$

$z' = r\cos\theta'$



Consider the first coordinate:

$$-\frac{1}{4\pi\epsilon_0}\iiint r\sin\theta'\cos\varphi'\,\nabla_x\rho(0)\,\frac{3a^2}{(r^2+a^2)^{5/2}}\,r^2dr\,\sin\theta' d\theta'\,d\varphi'$$

Here $\int_0^{2\pi} \cos\varphi' d\varphi' = 0$ and it cancels.

For y', the integral $\int_0^{2\pi} \sin\varphi' d\varphi' = 0$ will appear canceling this term as well

For z', $\int_0^{\pi} \cos\theta' \sin\theta' d\theta' = \frac{\sin^2\theta'}{2}\Big|_0^{\pi} = 0$ also.

Then, the linear term cancels entirely.

Consider now the second order term. First focus on the "crossed terms" involving

$x'y'$ or $x'z'$ or $y'z'$

For $x'y'$ we have the "angular" integral: $\int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \sin\theta' \cos\varphi' . \sin\theta' \sin\varphi' . \sin\theta'$

where $\int_0^{2\pi} d\varphi' \sin\varphi' \cos\varphi' = \frac{\sin^2\varphi'}{2}\Big|_0^{2\pi} = 0$

For $x'z'$ we have $\int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \sin\theta' \cos\varphi' . \cos\theta' . \sin\theta'$

and $\int_0^{2\pi} d\varphi' \cos\varphi' = 0$

For $y'z'$ we have $\int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \sin\theta' \sin\varphi' . \cos\theta' . \sin\theta'$

and $\int_0^{2\pi} d\varphi' \sin\varphi' = 0$

Then, only x'^2, y'^2, z'^2 may matter.

For x'^2 the angular integration is:

$$\int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \; \sin^2\theta' \cos^2\varphi' \; \sin\theta' = \int_0^{\pi} d\theta' \; \sin^3\theta' \cdot \int_0^{2\pi} \cos^2\varphi' d\varphi'$$

$$\cos 2\varphi' = \cos^2\varphi' - \sin^2\varphi'$$
$$= \cos^2\varphi' - (1-\cos^2\varphi') = 2\cos^2\varphi' - 1$$

$$\cos^2\varphi' = \frac{1+\cos 2\varphi'}{2}$$

$$\int_0^{2\pi} \cos^2\varphi' d\varphi' = \int_0^{2\pi} \left(\frac{1+\cos 2\varphi'}{2}\right) d\varphi' = \frac{1}{2} \cdot 2\pi = \pi$$

$$\int_0^{\pi} d\theta' \sin^3\theta' = \int_0^{\pi} d\theta' \; \sin\theta' (1-\cos^2\theta') d\theta' = -\cos\theta' \Big|_0^{\pi} + \frac{1}{3}\cos^3\theta' \Big|_0^{\pi}$$

$$= -(-1-1) + \frac{1}{3}(-1-1)$$

$$= 2 - \frac{2}{3} = \frac{4}{3}$$

Then, the full contribution of "x'^2" is:

$$-\frac{1}{4\pi\epsilon_0} \int_0^{R} \underbrace{\frac{1}{2} r^2 \left[\frac{\partial^2 \rho}{\partial x'^2}\right]}_{\text{From Taylor expansion}} \underbrace{\frac{4}{3}\pi}_{\substack{\text{From angular}\\ \text{integration}}} \frac{3a^2}{(r^2+a^2)^{5/2}} \underbrace{r^2 dr}_{\substack{\text{from } d^3x' \text{ after}\\ \text{integrating } \theta' \text{ and } \varphi'}}$$

evaluated at 0

$$= -\frac{1}{\epsilon_0} \int_0^{R} \frac{3a^2}{(r^2+a^2)^{5/2}} \; \frac{1}{6} r^2 \frac{\partial^2 \rho}{\partial x'^2} \; r^2 dr$$

For y'^2 the angular integration is:

$$\int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \, \sin^2\theta' \sin^2\varphi' \sin\theta'$$

and $\displaystyle\int_0^{2\pi} d\varphi' \sin^2\varphi' = \int_0^{2\pi} d\varphi' \, \frac{1-\cos 2\varphi'}{2} = \pi$ like for x'^2

The $\int d\theta'$ integral is the same.

$$\cos 2\varphi' = \cos^2\varphi' - \sin^2\varphi'$$
$$= 1 - 2\sin^2\varphi'$$
$$\sin^2\varphi' = \frac{1-\cos 2\varphi'}{2}$$

Thus, the contribution of y'^2 is the same as x'^2 just replacing $\frac{\partial^2}{\partial x'^2}\rho$ by $\frac{\partial^2}{\partial y'^2}\rho$!

For z'^2 the angular integration is:

$$\int_0^{\pi} d\theta' \int_0^{2\pi} d\varphi' \, \cos^2\theta' \sin\theta' = 2\pi \int_0^{\pi} d\theta' \cos^2\theta' \sin\theta'$$
$$= 2\pi \left(-\frac{\cos^3\theta'}{3}\right)\Bigg|_0^{\pi} = \left(\frac{-2\pi}{3}\right)(-1-1)$$
$$= \frac{4\pi}{3} \text{ like before!}$$

Thus the contribution of z'^2 is the same as x'^2 replacing $\frac{\partial^2}{\partial x'^2}\rho$ by $\frac{\partial^2}{\partial z'^2}\rho$.

Putting all together, the second order contribution in the Taylor expansion is:

$$-\frac{1}{\epsilon_0}\int_0^R \frac{3a^2}{(r^2+a^2)^{5/2}}\,\frac{1}{6}\,r^2\underbrace{\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}\right)}_{\nabla^2}\rho \quad r^2\,dr$$

which is the result in the book page 35.