11.2 Point Charges

11.2.1 Power Radiated by a Point Charge

In Chapter 10 we derived the fields of a point charge q in arbitrary motion (Eqs. 10.65 and 10.66):

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})], \tag{11.62}$$

where $\mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$, and

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t). \tag{11.63}$$

The first term in Eq. 11.62 is called the **velocity field**, and the second one (with the triple cross-product) is called the **acceleration field.**

The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{1}{\mu_0 c}[\mathbf{E} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E})] = \frac{1}{\mu_0 c}[E^2\hat{\boldsymbol{\mathscr{r}}} - (\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{E})\mathbf{E}]. \tag{11.64}$$

However, not all of this energy flux constitutes *radiation*; some of it is just field energy carried along by the particle as it moves. The *radiated* energy is the stuff that, in effect, *detaches* itself from the charge and propagates off to infinity. (It's like flies breeding on a garbage truck: Some of them hover around the truck as it makes its rounds; others fly away and never come back.) To calculate the total power radiated by the particle at time t_r, we draw a huge sphere of radius $\mathscr{r}$ (Fig. 11.11), centered at the position of the particle (at time t_r), wait the appropriate interval

$$t - t_r = \frac{\mathscr{r}}{c} \tag{11.65}$$

for the radiation to reach the sphere, and at that moment integrate the Poynting vector over the surface.[6] I have used the notation t_r because, in fact, this *is* the retarded time for all points on the sphere at time t.

Now, the area of the sphere is proportional to $\mathscr{r}^2$, so any term in $\mathbf{S}$ that goes like $1/\mathscr{r}^2$ will yield a finite answer, but terms like $1/\mathscr{r}^3$ or $1/\mathscr{r}^4$ will contribute nothing in the limit $\mathscr{r} \to \infty$. For this reason only the *acceleration* fields represent true radiation (hence their other name, **radiation fields**):

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}[\boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})]. \tag{11.66}$$

[6]Note the subtle change in strategy here: In Sect. 11.1 we worked from a fixed point (the origin), but here it is more appropriate to use the (moving) location of the charge. The implications of this change in perspective will become clearer in a moment.

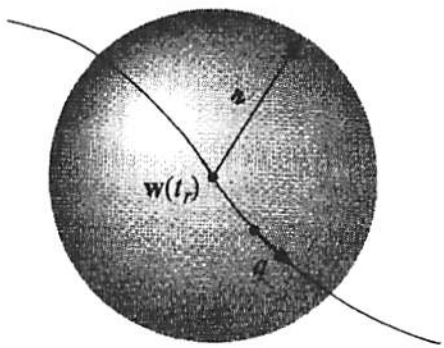

Figure 11.11

The velocity fields carry energy, to be sure, and as the charge moves this energy is dragged along—but it's not *radiation*. (It's like the flies that stay with the garbage truck.) Now $\mathbf{E}_{\text{rad}}$ is perpendicular to $\hat{\boldsymbol{\mathscr{r}}}$, so the second term in Eq. 11.64 vanishes:

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c} E_{\text{rad}}^2 \,\hat{\boldsymbol{\mathscr{r}}}. \tag{11.67}$$

If the charge is instantaneously at *rest* (at time t_r), then $\mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}}$, and

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c^2 \mathscr{r}}[\hat{\boldsymbol{\mathscr{r}}} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{a})] = \frac{\mu_0 q}{4\pi \mathscr{r}}[(\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{a})\hat{\boldsymbol{\mathscr{r}}} - \mathbf{a}]. \tag{11.68}$$

In that case

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi \mathscr{r}}\right)^2 [a^2 - (\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{a})^2]\,\hat{\boldsymbol{\mathscr{r}}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\mathscr{r}^2}\right)\hat{\boldsymbol{\mathscr{r}}}, \tag{11.69}$$

where θ is the angle between $\hat{\boldsymbol{\mathscr{r}}}$ and $\mathbf{a}$. No power is radiated in the forward or backward direction—rather, it is emitted in a donut about the direction of instantaneous acceleration (Fig. 11.12).

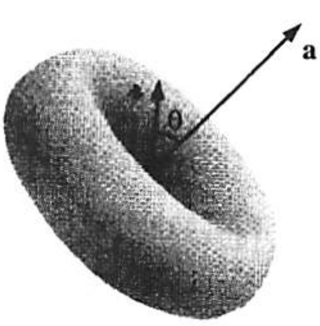

Figure 11.12

The total power radiated is evidently

$$P = \oint \mathbf{S}_{\text{rad}} \cdot d\mathbf{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int \frac{\sin^2\theta}{\mathcal{r}^2} \mathcal{r}^2 \sin\theta \, d\theta \, d\phi,$$

or

$$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}.} \tag{11.70}$$

This, again, is the **Larmor formula**, which we obtained earlier by another route (Eq. 11.61).

Although I *derived* them on the assumption that $v = 0$, Eqs. 11.69 and 11.70 actually hold to good approximation as long as $v \ll c$. An exact treatment of the case $v \neq 0$ is more difficult,[7] both for the obvious reason that $\mathbf{E}_{\text{rad}}$ is more complicated, and also for the more subtle reason that $\mathbf{S}_{\text{rad}}$, the rate at which energy passes through the sphere, is *not* the same as the rate at which energy left the particle. Suppose someone is firing a stream of bullets out the window of a moving car (Fig.11.13). The rate N_t at which the bullets strike a stationary target is not the same as the rate N_g at which they left the gun, because of the motion of the car. In fact, you can easily check that $N_g = (1 - v/c)N_t$, if the car is moving towards the target, and

$$N_g = \left(1 - \frac{\hat{\mathcal{r}} \cdot \mathbf{v}}{c}\right) N_t$$

for arbitrary directions (here $\mathbf{v}$ is the velocity of the car, c is that of the bullets—relative to the ground—and $\hat{\mathcal{r}}$ is a unit vector from car to target). In our case, if dW/dt is the rate at which energy passes through the sphere at radius $\mathcal{r}$, then the rate at which energy left the charge was

$$\frac{dW}{dt_r} = \frac{dW/dt}{\partial t_r / \partial t} = \left(\frac{\boldsymbol{\mathcal{r}} \cdot \mathbf{u}}{\mathcal{r} c}\right) \frac{dW}{dt}. \tag{11.71}$$

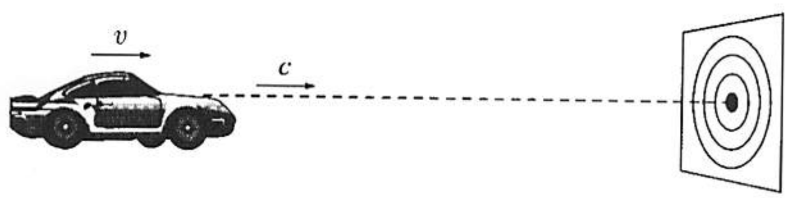

Figure 11.13

[7] In the context of special relativity, the condition $v = 0$ simply represents an astute choice of reference system, with no essential loss of generality. If you can decide how P transforms, you can *deduce* the general (Liénard) result from the $v = 0$ (Larmor) formula (see Prob. 12.69).

(I used Eq. 10.71 to express $\partial t_r/\partial t$.) But

$$\frac{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}{\mathscr{r}c} = 1 - \frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{v}}{c},$$

which is precisely the ratio of N_g to N_t; it's a purely geometrical factor (the same as in the Doppler effect).

The power radiated by the particle into a patch of area $\mathscr{r}^2 \sin\theta \, d\theta \, d\phi = \mathscr{r}^2 \, d\Omega$ on the sphere is therefore given by

$$\frac{dP}{d\Omega} = \left(\frac{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}{\mathscr{r}c}\right)\frac{1}{\mu_0 c}E_{\text{rad}}^2 \mathscr{r}^2 = \frac{q^2}{16\pi^2\epsilon_0}\frac{|\hat{\boldsymbol{\mathscr{r}}}\times(\mathbf{u}\times\mathbf{a})|^2}{(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{u})^5}, \tag{11.72}$$

where $d\Omega = \sin\theta \, d\theta \, d\phi$ is the **solid angle** into which this power is radiated. Integrating over θ and ϕ to get the total power radiated is no picnic, and for once I shall simply quote the answer:

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\mathbf{v}\times\mathbf{a}}{c}\right|^2\right), \tag{11.73}$$

where $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$. This is **Liénard's generalization** of the Larmor formula (to which it reduces when $v \ll c$). The factor γ^6 means that the radiated power increases enormously as the particle velocity approaches the speed of light.

Example 11.3

Suppose $\mathbf{v}$ and $\mathbf{a}$ are instantaneously collinear (at time t_r), as, for example, in straight-line motion. Find the angular distribution of the radiation (Eq. 11.72) and the total power emitted.

Solution: In this case $(\mathbf{u}\times\mathbf{a}) = c(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{a})$, so

$$\frac{dP}{d\Omega} = \frac{q^2c^2}{16\pi^2\epsilon_0}\frac{|\hat{\boldsymbol{\mathscr{r}}}\times(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{a})|^2}{(c-\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{v})^5}.$$

Now

$$\hat{\boldsymbol{\mathscr{r}}}\times(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{a}) = (\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a})\hat{\boldsymbol{\mathscr{r}}} - \mathbf{a}, \quad \text{so } |\hat{\boldsymbol{\mathscr{r}}}\times(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{a})|^2 = a^2 - (\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a})^2.$$

In particular, if we let the z axis point along $\mathbf{v}$, then

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}, \tag{11.74}$$

where $\beta \equiv v/c$. This is consistent, of course, with Eq. 11.69, in the case $v = 0$. However, for very large v ($\beta \approx 1$) the donut of radiation (Fig. 11.12) is stretched out and pushed forward by the factor $(1-\beta\cos\theta)^{-5}$, as indicated in Fig. 11.14. Although there is still no radiation in *precisely* the forward direction, most of it is concentrated within an increasingly narrow cone *about* the forward direction (see Prob. 11.15).

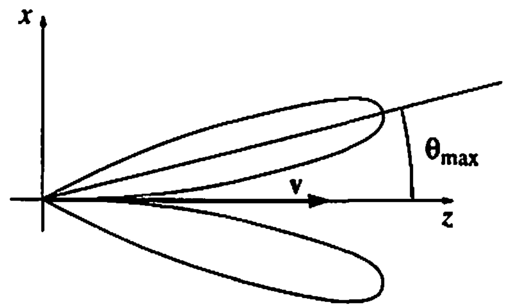

Figure 11.14

The *total* power emitted is found by integrating Eq. 11.74 over all angles:

$$P = \int \frac{dP}{d\Omega}\, d\Omega = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \sin\theta\, d\theta\, d\phi.$$

The ϕ integral is 2π; the θ integral is simplified by the substitution $x \equiv \cos\theta$:

$$P = \frac{\mu_0 q^2 a^2}{8\pi c} \int_{-1}^{+1} \frac{(1-x^2)}{(1-\beta x)^5}\, dx.$$

Integration by parts yields $\frac{4}{3}(1-\beta^2)^{-3}$, and I conclude that

$$P = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c}. \tag{11.75}$$

This result is consistent with the Liénard formula (Eq. 11.73), for the case of collinear **v** and **a**. Notice that the angular distribution of the radiation is the same whether the particle is *accelerating* or *decelerating*; it only depends on the *square* of a, and is concentrated in the forward direction (with respect to the velocity) in either case. When a high speed electron hits a metal target it rapidly decelerates, giving off what is called **bremsstrahlung**, or "braking radiation." What I have described in this example is essentially the classical theory of bremsstrahlung.

Problem 11.13

(a) Suppose an electron decelerated at a constant rate a from some initial velocity v_0 down to zero. What fraction of its initial kinetic energy is lost to radiation? (The rest is absorbed by whatever mechanism keeps the acceleration constant.) Assume $v_0 \ll c$ so that the Larmor formula can be used.

(b) To get a sense of the numbers involved, suppose the initial velocity is thermal (around 10^5 m/s) and the distance the electron goes is 30 Å. What can you conclude about radiation losses for the electrons in an ordinary conductor?

Problem 11.14 In Bohr's theory of hydrogen, the electron in its ground state was supposed to travel in a circle of radius 5×10^{-11}m, held in orbit by the Coulomb attraction of the proton.

According to classical electrodynamics, this electron should radiate, and hence spiral in to the nucleus. Show that $v \ll c$ for most of the trip (so you can use the Larmor formula), and calculate the lifespan of Bohr's atom. (Assume each revolution is essentially circular.)

Problem 11.15 Find the angle $\theta_{\max}$ at which the maximum radiation is emitted, in Ex. 11.3 (see Fig. 11.14). Show that for ultrarelativistic speeds (v close to c), $\theta_{\max} \cong \sqrt{(1-\beta)/2}$. What is the intensity of the radiation in this maximal direction (in the ultrarelativistic case), in proportion to the same quantity for a particle instantaneously at rest? Give your answer in terms of γ.

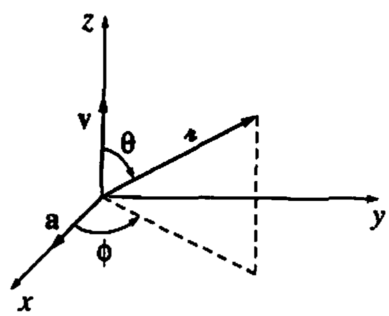

Figure 11.15

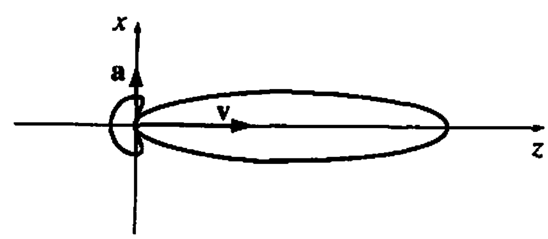

Figure 11.16

! **Problem 11.16** In Ex. 11.3 we assumed the velocity and acceleration were (instantaneously, at least) *collinear*. Carry out the same analysis for the case where they are *perpendicular*. Choose your axes so that $\mathbf{v}$ lies along the z axis and $\mathbf{a}$ along the x axis (Fig. 11.15), so that $\mathbf{v} = v\,\hat{\mathbf{z}}$, $\mathbf{a} = a\,\hat{\mathbf{x}}$, and $\hat{\boldsymbol{\imath}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$. Check that P is consistent with the Liénard formula. [*Answer:*

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{[(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\phi]}{(1-\beta\cos\theta)^5}, \quad P = \frac{\mu_0 q^2 a^2 \gamma^4}{6\pi c}.$$

For relativistic velocities ($\beta \approx 1$) the radiation is again sharply peaked in the forward direction (Fig. 11.16). The most important application of these formulas is to *circular* motion—in this case the radiation is called **synchrotron radiation**. For a relativistic electron the radiation sweeps around like a locomotive's headlight as the particle moves.]

11.2.2 Radiation Reaction

According to the laws of classical electrodynamics, an accelerating charge radiates. This radiation carries off energy, which must come at the expense of the particle's kinetic energy. Under the influence of a given force, therefore, a charged particle accelerates *less* than a neutral one of the same mass. The radiation evidently exerts a force ($\mathbf{F}_{\text{rad}}$) back on the charge—a *recoil* force, rather like that of a bullet on a gun. In this section we'll derive the