Introduction to Electrostatics

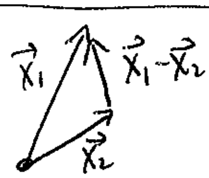
Phenomena involving time-independent distributions of charge and fields. It is mainly a review, this is why we will not discuss topics too much.

1.1 Coulomb's Law

Experimentally, it is known that:

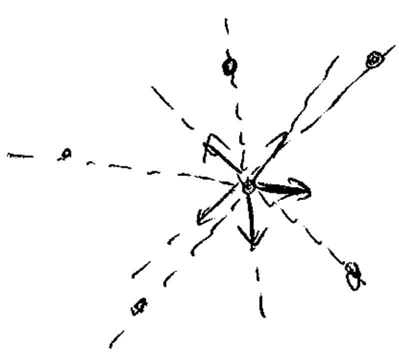
$$(1.2) \qquad \boxed{\vec{F} = k\, q_1 q_2 \frac{(\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|^3}}$$

Between two particles of charges q_1 and q_2, the force goes like the inverse square of the distance, along the line joining them.

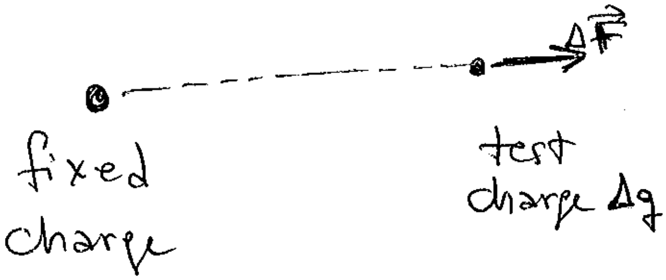


This is the basis of all that follows!

It is also known that the total force on a charge produced by many others is the vector sum of the individual two-body forces.



1.2 Electric Field



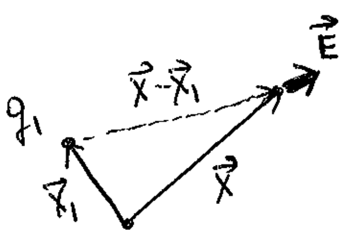
$$\boxed{\vec{E} = \lim_{\Delta q \to 0} \frac{\Delta \vec{F}}{\Delta q}}$$

The limit $\Delta q \to 0$ is taken so that the charge that we call "test" does not alter the distribution of charges and fields itself.

$$(1.2) \quad \boxed{\vec{F} = q\vec{E}}$$

; $\vec{E}(\vec{x})$ is a property of point $(\vec{x})$ even if q is not there. So $\vec{E}$ has a reality independent of q, and dependent on the other charges in the system.

For the case of Coulomb's law, then



$$\boxed{\vec{E}(\vec{x}) = k q_1 \frac{(\vec{x}-\vec{x}_1)}{|\vec{x}-\vec{x}_1|^3}} \quad (1.3)$$

is the electric field at $\vec{x}$ produced by charge q_1 located at $\vec{x}_1$.

"k" is a proportionality constant that depends on the system of units used. In the book of Jackson, the "SI" system is followed. (Système International d'Unités).

In SI, $k = \frac{1}{4\pi\epsilon_0} \left(= \frac{1}{10^{-7} C^2} \right)$

$$\boxed{\epsilon_0 \approx 8.854 \times 10^{-12} \, \frac{F}{m}}$$

F ← Farad ← unit of capacitance

m ← meter

(* See ~~next~~ extra page)

↑ "permittivity of free space".

underlined are the four fundamental units.

In SI: kg, second, meter, amper, volts, etc. a the units.

In SI:

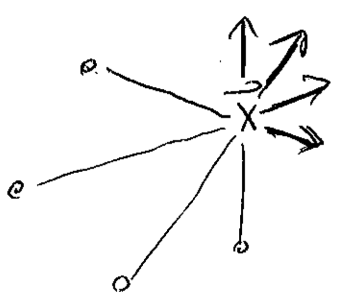
charge → Coulomb C

electric field → $\frac{V}{m}$ (volts per meter)

$\boxed{\text{See Table 4, page 783 for details}}$ ← read

Due to linear superposition of forces due to many charges, we have a similar formula for $\vec{E}$:

$$\boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{(\vec{x}-\vec{x}_i)}{|\vec{x}-\vec{x}_i|^3}} \qquad (1.4)$$



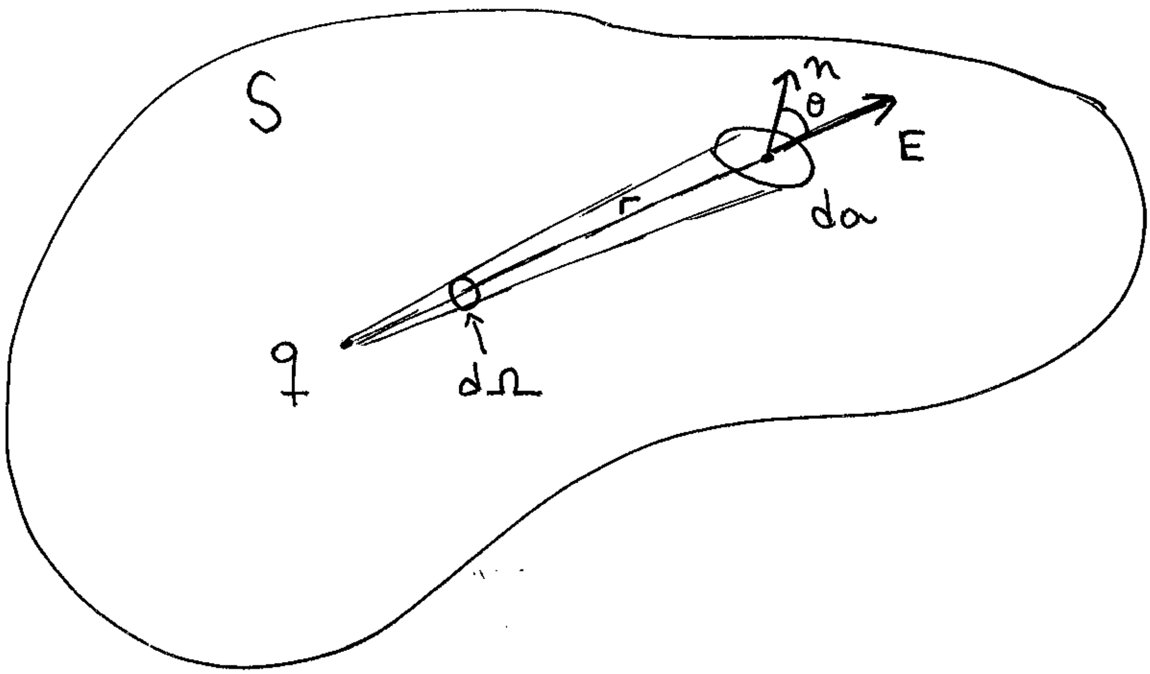
For a continuum distribution of charges:

$$\boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}') \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} \underbrace{d^3x'}_{dx'dy'dz'}} \qquad (1.5)$$

↑ Useful formula, but not always the best to use, so in 1.3 and beyond we will learn of many other ways to get a $\vec{E}$. (see?)

1.3 Gauss's Law ← often more useful than (1.5).

Consider a point charge q and a closed surface S:



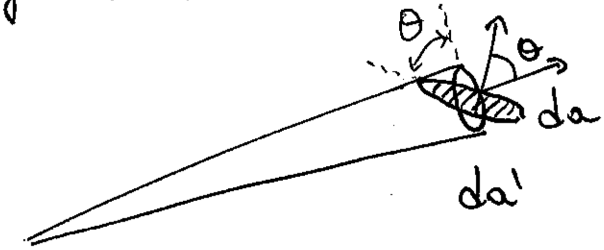
$\vec{n}$ is a unit vector normal to the surface "outwardly" directed

da is an element of surface area

$$\text{Then} \quad (\vec{E}\cdot\vec{n})\, da = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r^2}\cdot\underbrace{|\vec{n}|}_{1}\cdot\cos\theta \, da \qquad (1.7)$$

Moreover, $r^2 d\Omega$ would be the element of surface area if $\vec{n}$ would point along $\vec{E}$. If not, we must correct for the fact that da is inclined by θ. Then:

$$da = \frac{r^2 d\Omega}{\cos\theta}$$

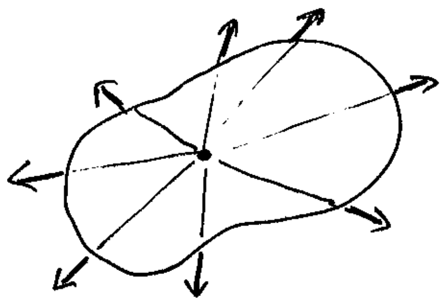


Then, $$(\vec{E}\cdot\vec{n})\, da = \frac{q}{4\pi\epsilon_0}\, d\Omega \qquad (1.8)$$

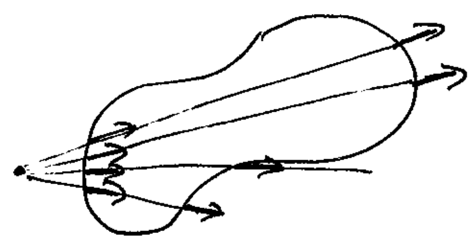
and, integrating,

$$\oint_S (\vec{E}\cdot\vec{n})\, da = \frac{q}{4\pi\epsilon_0} \oint_S d\Omega$$

Note that q could have been inside S, in which case all contributions of $(\vec{E}\cdot\vec{n})$ are positive, or outside and then some are positive others negative:

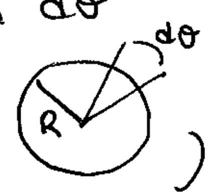


(a)



(b)

Since $d\Omega$ is <u>independent</u> of r (it is like a $d\theta$ in a circle which is indep. of R



)

then

$$\oint_S d\Omega = 4\pi \quad \text{if } q \text{ inside } S$$

and $$\oint_S d\Omega = 0$$ if q outside, since each line in the figure (b) crosses <u>twice</u> the surface, thus cancelling the contributions.

Then:

$$\boxed{\oint_S (\vec{E}\cdot\vec{m})\, da = \begin{cases} q/\epsilon_0 & \text{if } q \text{ } \cancel{\text{out}} \text{ inside } S \\ 0 & \text{if } q \text{ outside } S \end{cases}} \qquad (1.9)$$

Generalized to many charges

$$\oint_S (\vec{E}\cdot\vec{n})\, da = \frac{1}{\epsilon_0} \sum_i q_i \qquad (1.10)$$

i ← inside S

(due to linear superposition)

or, for a continuum:

$$\boxed{\oint_S (\vec{E}\cdot\vec{n})\, da = \frac{1}{\epsilon_0} \int_V \rho(\vec{x})\, d^3x} \qquad (1.11)$$

↑ Gauss's law

This is one of the <u>basic</u> equations of electrostatics.

It depends on the "$\frac{1}{r^2}$" law of the Coulomb force, since the $\frac{1}{r^2}$ cancelled the geometric r^2 coming from da.
If the law would have been say $\frac{1}{r^{2+\epsilon}}$, then there would be no Gauss's law.

We are also using "superposition" to apply this law.

THUS, this same law also holds for Newtonian gravity

1.4 Differential form of Gauss's Law

Simply use the "divergence theorem"

$$\oint_S (\vec{A}.\vec{n})\, da = \int_V (\nabla.\vec{A})\, d^3x \qquad (\vec{n} \text{ points outwardly})$$

which is valid for any "well behaved" vector field $\vec{A}(\vec{x})$. V is the volume surrounded by S, and S is closed.

Applied to Eq. (1.11), we get:

$$\oint_S (\vec{E}.\vec{n})\, da = \int_V (\nabla.\vec{E})\, d^3x = \frac{1}{\epsilon_0}\int_V \rho(\vec{x})\, d^3x$$

Since this is true for any (S, V) then:

$$\boxed{\nabla.\vec{E}(\vec{x}) = \frac{\rho(\vec{x})}{\epsilon_0}}$$

which is the differential form of Gauss's law of electrostatics.

This eq. can be used to solve problems, but we can go one step further and define a scalar since often it is simpler to deal with them than with vector fields

1.5 Scalar Potential

From the equation $\nabla \cdot \vec{E} = \rho/\epsilon_0$ we cannot totally specify $\vec{E}$. We need also an equation involving $\nabla \times \vec{E}$. (and even this is only "almost" enough)

Consider

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{x}\,') \underbrace{\frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3}}_{\text{This is } -\nabla\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)} d^3x' \quad (1.5)$$

Volume large enough to include all charges.

Then:

$$\vec{E}(\vec{x}) = -\frac{1}{4\pi\epsilon_0} \nabla \underbrace{\int \frac{\rho(\vec{x}\,')\, d^3x'}{|\vec{x}-\vec{x}\,'|}}_{f(\vec{x})}$$

Take $\nabla \times$ on both sides:

$$\nabla \times \vec{E}(\vec{x}) = -\frac{1}{4\pi\epsilon_0} \left[\nabla \times \nabla f(\vec{x}) \right]$$

and the curl of a gradient vanishes for any well behaved $f(\vec{x})$. Then:

$$\boxed{\nabla \times \vec{E}(\vec{x}) = 0}$$

Let us go back a few steps and write:

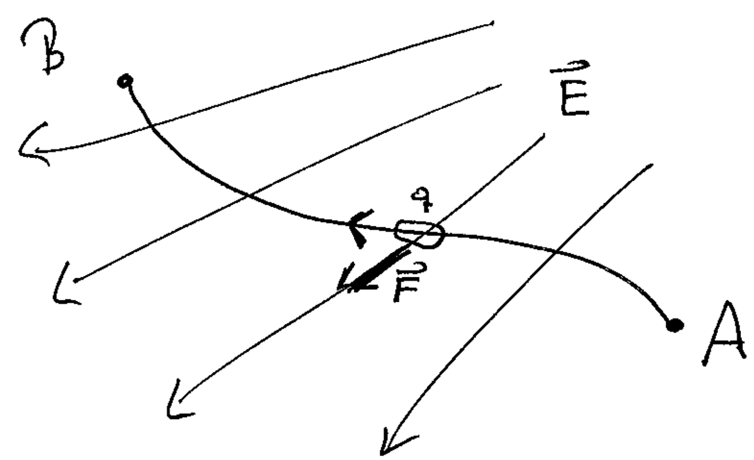
$$\vec{E}(\vec{x}) = -\nabla\left[\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}\right]$$

The function [...] is a <u>scalar</u>. It is called the scalar potential $\Phi(\vec{x})$. It is usually easier to calculate scalar functions than vector functions, thus $\Phi(\vec{x})$ it is often used as an intermediate step to get $\vec{E}$.

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}}$$

integrated over all charges, i.e. over a V large enough to include all of $\rho(\vec{x}')$

$\Phi(\vec{x})$ acquires a physical meaning when we consider the <u>work</u> done on a test charge q to move it from point A to B, in the presence of an electric field $\vec{E}(\vec{x})$



$$\underset{A\to B}{\text{Work}} = -\int_A^B \overrightarrow{\text{Force}} \cdot \vec{dl} = -q\int_A^B \vec{E}\cdot\vec{dl} =$$

$$= q\int_A^B \underbrace{\nabla\Phi\cdot\vec{dl}}_{\substack{\text{like } \frac{d\Phi}{dx}\cdot dx \\ \text{in one dimension}}} = q\int_A^B d\Phi = \boxed{q(\Phi_B-\Phi_A) = W}$$

Then, $q\Phi$ can be interpreted as a <u>potential energy</u> of the 'test charge "q" in the electrostatic field.

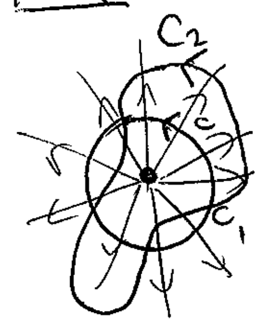
Note also that

$$\int_A^B \vec{E}\cdot\vec{dl} = -(\Phi_A-\Phi_B)$$ is <u>independent of the path</u>.

Then, if the path is closed:

$$\boxed{\oint_{\substack{\text{closed}\\\text{path}}} \vec{E}\cdot\vec{dl} = 0}$$

<u>Example</u>

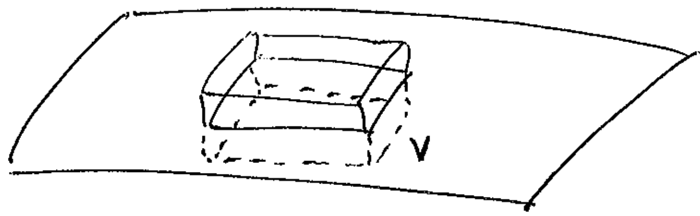


$$\oint_{C_1} \vec{E}\cdot\vec{dl} = 0$$

$$\oint_{C_2} \vec{E}\cdot\vec{dl} =$$

1.6 Surface distribution of charges and dipoles

A "surface distribution of charges" appears often in problems of electrostatics.



From
$$\oint_S \vec{E}.\vec{m}\, da = \frac{1}{\epsilon_0}\int_V \rho(\vec{x})\, d^3x$$
(Gauss's law)

$\underbrace{\int \rho(\vec{x})\, d^3\vec{x}}_{\text{charge enclosed}} = \int_S \sigma(\vec{x})\, da$ Thus:

$\oint_S ((\vec{E}_2 - \vec{E}_1).\vec{n})\, da = \frac{1}{\epsilon_0}\int \sigma(\vec{x})\, da$

~~or else~~

$\cancel{\rho(\vec{x}) = \sigma(\vec{x})\,\delta(z)}$

~~z goes ⊥ to surface~~

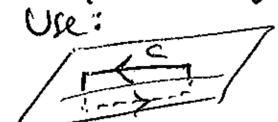
we get
$$\boxed{(\vec{E}_2 - \vec{E}_1).\vec{n} = \frac{1}{\epsilon_0}\sigma(\vec{x})}$$

surface charge density

This shows that the electric field (actually the normal component of $\vec{E}$) is discontinuous in crossing a surface charge density.

The tangential component is continuous (left as exercise).

Use:



and (1.21)
$\oint_C \vec{E}.d\vec{l} = 0$

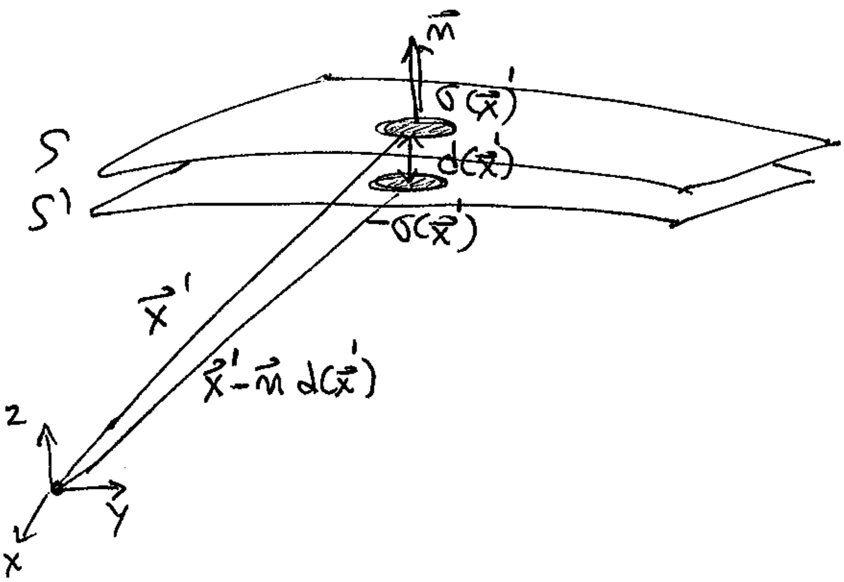
The potential caused by $\sigma(\vec{x})$ is:

If $\sigma(x) = \sigma_0$
$E =$ constant

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'|}\, da'}$$

This potential is continuous even with the surface distrib. of charge.

Potential due to a dipole layer



$S' \to S$

while $\sigma \to \infty$

in such a way that $\lim_{d(\vec{x}) \to 0} \sigma(\vec{x})\, d(\vec{x}) = D(\vec{x})$ is finite

$\vec{D}$ as vector pos from (−) to (+) charge.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\vec{x}')\,da'}{|\vec{x}-\vec{x}'|} - \frac{1}{4\pi\epsilon_0} \int_{S'} \frac{\sigma(\vec{x}')\,da'}{|\vec{x}-\vec{x}'+\vec{n}\,d(\vec{x}')|}$$

Let us expand using

$$\frac{1}{|\vec{x}+\vec{a}|} = \frac{1}{|\vec{x}|} + \vec{a}\cdot\nabla\left(\frac{1}{|\vec{x}|}\right) + \cdots \quad \text{valid if } |\vec{a}| \text{ is small.}$$

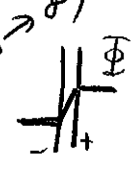
Using $\vec{a} = \vec{n}\,d(\vec{x}')$ we get:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_S \underbrace{\sigma(\vec{x}')\,\vec{n}\,d(\vec{x}')}_{D(\vec{x}')} \cdot \nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) da'$$

It can be shown to be discontinuous

Critical to $\sigma \to \infty$)

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_S D(\vec{x}')\; \vec{n}\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) da'} \quad (1.24)$$



1.7 Poisson and Laplace equations

$$\nabla \cdot \vec{E} = \rho/\epsilon_0$$

$$\nabla \times \vec{E} = 0 \implies \vec{E} = -\nabla \Phi$$

Then $\nabla \cdot (-\nabla \Phi) = \rho/\epsilon_0$

or $$\boxed{\nabla^2 \Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0}}$$ "Poisson equation"

If $\rho(\vec{x}) = 0$, then we get the "Laplace equation":

$$\nabla^2 \Phi(\vec{x}) = 0 .$$

We already know that

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} d^3x'$$

In a homework problem we will do this a bit better mathematically.

from previous sections. If this $\Phi(\vec{x})$ should satisfy Poisson's equations, then:

$$\nabla^2 \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \underbrace{\rho(\vec{x}\,')}_{\substack{\nabla \text{ affects} \\ \vec{x},\text{ not } \vec{x}\,'!}} \nabla^2 \left(\frac{1}{|\vec{x}-\vec{x}\,'|} \right) d^3x'$$

For the r.h.s. to be $\frac{\rho}{\epsilon_0}$ we need $$\boxed{\nabla^2 \frac{1}{|\vec{x}-\vec{x}\,'|} = -4\pi\delta(\vec{x}-\vec{x}\,')}$$

1.8 Green's Theorem

If the problems of electrostatic would always involve localized (discrete or continuous) distributions of charge, with no boundary effects, then $\Phi(x) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x')}{|x-x'|} d^3x'$ would be sufficient. However, most problems involve finite regions with particular boundary conditions.

Let us first deduce some identities:

$$\int_V \nabla\cdot\vec{A}\, d^3x = \oint_S \vec{A}\cdot\vec{n}\, da$$

$\vec{A}$ "well behaved"

Consider $\vec{A} = \phi\nabla\psi$ (ϕ, ψ arbitrary scalar functions)

The Green theorem provides a procedure to handle boundary problems.

$$\nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$$

$$\vec{A}\cdot\vec{n} = \phi\nabla\psi\cdot\vec{n} = \phi\,\underbrace{\text{"}\frac{\partial\psi}{\partial n}\text{"}}$$

definition of symbol "$\frac{\partial}{\partial n}$"

normal derivative at the surface (outwards)

↳ i.e. from the V considered to the outside

Then:

$$\int_V \underbrace{[\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi]}_{\nabla\cdot(\phi\nabla\psi)}\, d^3x = \oint_S \phi\frac{\partial\psi}{\partial n}\, da \qquad (1.34)$$

Interchanging: $\phi \leftrightarrow \psi$

$$\int_V [\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi]\, d^3x = \oint_S \psi\frac{\partial\phi}{\partial n}\, da$$

Taking the difference:

$$\int_V \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) d^3x = \oint_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) da$$

which is "Green's theorem". The idea is to replace the Poisson equation by an integral equation:

diff.

Now consider the special case:

$$\psi = \frac{1}{|\underset{\text{observation point}}{\vec{x}} - \underset{\text{integration variable}}{\vec{x}'}|} \quad \text{and} \quad \phi = \Phi(\vec{x}) \leftarrow \text{scalar potential}$$

and use $\nabla^2 \Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0}$

Then:

$$\int_V \left[\Phi(\vec{x}') \underbrace{\nabla^2_{x'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)}_{\substack{-4\pi\delta(\vec{x}-\vec{x}') \\ (1.31)}} - \frac{1}{|\vec{x}-\vec{x}'|}\left(\frac{-\rho(\vec{x}')}{\epsilon_0}\right)\right] d^3x' =$$

$$= -4\pi \underbrace{\Phi(\vec{x})}_{\substack{\text{Note that } \vec{x} \text{ must} \\ \text{be inside } V \text{ for this to be true!}}} + \int_V \frac{\rho(\vec{x}')\, d^3x'}{\epsilon_0 |\vec{x}-\vec{x}'|}$$

$$= \oint_S \left[\Phi \frac{\partial}{\partial n'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) - \frac{1}{|\vec{x}-\vec{x}'|} \underbrace{\frac{\partial \Phi}{\partial n'}}\right] da'$$

This symbol with "'" means $\nabla' \Phi(\vec{x}') \cdot \vec{n}$

Then:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}\,')\,d^3x'}{|\vec{x}-\vec{x}\,'|} + \frac{1}{4\pi}\oint_S \left[\frac{1}{|\vec{x}-\vec{x}\,'|}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)\right]da' \qquad (1.36)$$

$\Phi(\vec{x})$: observation point inside V

This can be rewritten as:

$$\frac{1}{4\pi\epsilon_0}\oint_S \frac{1}{|\vec{x}-\vec{x}\,'|}\underbrace{\epsilon_0\frac{\partial\Phi}{\partial n'}}_{\equiv\,\sigma(\vec{x}\,')}da'$$

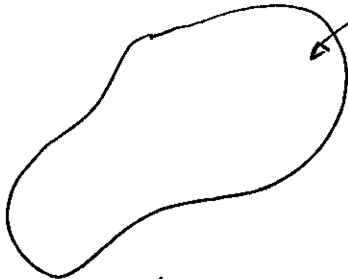
and we recover (1.23) (see (I.17))

If $D = -\epsilon_0\Phi$ then this term becomes (1.24) which is the potential caused by a "surface dipole moment" $D(\vec{x})$ (see also 1.26) $\left[\vec{n}\cdot\nabla'\frac{1}{|\vec{x}-\vec{x}\,'|} \equiv \frac{\partial}{\partial n'}\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)\right]$

So this equation appears to make sense!
If we know Φ (and $\frac{\partial\Phi}{\partial n'}$) at the surface, then with $\rho(\vec{x})$ also known, ($\sigma(\vec{x})$ and $D(\vec{x})$) we can get the entire $\Phi(\vec{x})$ inside.
If $\rho(\vec{x}) \equiv 0$ inside V, then just Φ_{surface} and $\frac{\partial\Phi}{\partial n}$ surface is sufficient to know $\Phi(\vec{x})$ in the entire volume.

<u>HOWEVER</u>: In practice, <u>only</u> Φ or $\frac{\partial\Phi}{\partial n}$ is known at a surface, <u>not</u> both simultaneously. Thus, the equation above for Φ is <u>not</u> useful. And I <u>cannot</u> specify arbitrary values for Φ and $\frac{\partial\Phi}{\partial n}$ at the surface since they must be linked.

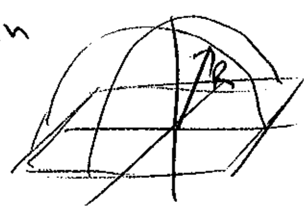
1.9 Dirichlet or Neumann Boundary Condition



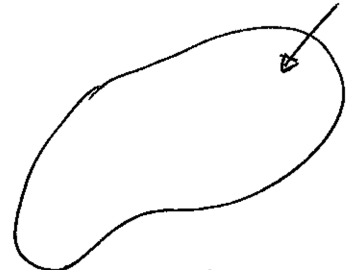
If I know Φ in this surface, the Φ is unique all over.

"Dirichlet" boundary condition is when we specify the potential on a closed surface

(note that this closed surface may include portions at "∞" such as in



$R \to \infty$ for a half sphere.



If I know $\vec{E}$ in this surface, I can solve the problem as well.

"Neumann" boundary condition is when we specify the electric field (i.e. $\frac{\partial \Phi}{\partial n}$) everywhere on the closed surface

We will not prove it, but Sec. 1.9 shows that with these boundary conditions the solution is unique.

[Note that in general we should not mix these two boundary conditions i.e. at the surface Φ and $\frac{\partial \Phi}{\partial n}$ of course must be related so I cannot assign arbitrary values to them. Fix one or the other and work out the problem.]

1.10 Green Functions and value problems

We will discuss a method to solve the Laplace or Poisson eq. with boundary conditions on a surface S

By definition, Green functions satisfy

$$\nabla^2_{x'} G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}') \cdot \quad (1.31)$$

Instead of solving a differential equation we will solve an integral equation!

(So $\frac{1}{|\vec{x}-\vec{x}'|}$ is a special case of Green function)

In general,

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}') \quad (1.40)$$

where $F(\vec{x},\vec{x}')$ satisfies $\nabla^2_{x'} F(\vec{x},\vec{x}') = 0$ (1.41) inside the volume V we are studying.

F may not satisfy $\nabla^2 F = 0$ outside but that is not a problem.

Let us go back to Green's theorem (1.35) and use $\varphi = \Phi$ and $\psi = G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F$. Then:

~~$\Phi(\vec{x}) =$~~

$$\int_V \Big(\Phi \underbrace{\nabla'^2 G}_{-4\pi\delta(\vec{x}-\vec{x}')} - G \underbrace{\nabla'^2 \Phi}_{-\rho/\epsilon_0} \Big) d^3x' = \oint_S \left(\Phi \frac{\partial G}{\partial n'} - G \frac{\partial \Phi}{\partial n'} \right) da$$

$$-4\pi\,\Phi(\vec{x}) + \frac{1}{\epsilon_0}\int_V \rho(\vec{x}')\,G(\vec{x},\vec{x}')\,d^3x' = \oint_S \left(\Phi \frac{\partial G}{\partial n'} - G \frac{\partial \Phi}{\partial n'} \right) da'$$

or

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\,G(\vec{x},\vec{x}')\,d^3x' + \frac{1}{4\pi}\oint_S \left(G\frac{\partial \Phi}{\partial n'} - \Phi\frac{\partial G}{\partial n'} \right) da' \quad (1.42)$$

For a Dirichlet boundary condition we demand $G_D(\vec{x},\vec{x}')=0$ for $\vec{x}'$ on S. We have the freedom of "demanding" properties because we can still choose $F(\vec{x},\vec{x}')$ as we wish so long as $\nabla'^2 F = 0$ inside V.

Then: one term, the one with G_D, at S cancels

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\, G_D(\vec{x},\vec{x}')\, d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\, \frac{\partial G_D}{\partial n'}\, da'}$$

where $G_D(\vec{x},\vec{x}')=0$ for $\vec{x}'$ on S

(Neumann's b. cond. are left to exercise)

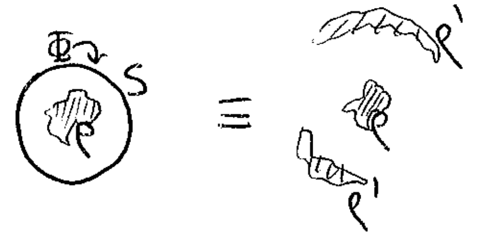
If $G_D(\vec{x},\vec{x}')$ can be found somehow, and we will have at least one problem in the homework doing so, then with the knowledge of $\Phi(\vec{x}')$ at the surface S, we can get $\Phi(\vec{x})$ in all V. Note that $G_D(\vec{x},\vec{x}')$ only depends on the shape of S, namely it can be used the same G_D for any $\Phi(\vec{x}')$ at the surface S and any $\rho(\vec{x}')$ inside the volume.

So $G_D(\vec{x},\vec{x}')$ is a property of the geometry of the problem.

It can be shown that

$$G(\vec{x},\vec{x}') = G(\vec{x}',\vec{x}).$$

Also, since $\nabla'^2 F(\vec{x},\vec{x}')=0$, then $F(\vec{x},\vec{x}')$ is the solution of Laplace's equation with sources outside V. Thus, to get a $G_D(\vec{x},\vec{x}')=0$ at S, we have to select a set of charges outside V, such that the boundary condition is satisfied.



such that ρ' gives Φ at S.

|| There will be one HW problem in Chapter 2 about this subject. ||

1.11 Electrostatic Potential Energy and Energy Density; Capacitance

In section 1.6, it was already shown that if a point charge q is brought from infinity to $\vec{x}$ in a region described by a potential $\Phi(\vec{x})$, then the work done on the charge (potential energy) is

$$W_i = q_i \, \Phi(\vec{x}_i)$$

"i" is the label of the charge q_i

Since the potential is

$$\Phi(\vec{x}_i) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\vec{x}_i - \vec{x}_j|}$$

Assuming all charges are just a group of n discrete ones. i.e. localized charges

produced by the other $n-1$ charges (assumed fixed), then

$$W_i = \frac{q_i}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\vec{x}_i - \vec{x}_j|}.$$

The total energy

$$\boxed{W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}} \qquad (1.50)$$

adding one charge at a time.

or

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ (i \neq j)}}^{n} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \qquad (1.51)$$

For a continuous distribution

$$W = \frac{1}{8\pi\epsilon_0} \int\int \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \, d^3x \, d^3x' \qquad (1.52)$$

$$= \frac{1}{2} \int \rho(\vec{x}) \Phi(\vec{x}) \, d^3x \qquad (1.53)$$

since for a localized set of charges

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \, d^3x'$$

An alternative expression uses $\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$

$$W = -\frac{\epsilon_0}{2} \int \Phi \nabla^2 \Phi \, d^3x = \frac{\epsilon_0}{2} \int |\nabla \Phi|^2 d^3x =$$

$$= \frac{\epsilon_0}{2} \int |\vec{E}(\vec{x})|^2 d^3x \qquad (1.54)$$

$\vec{E} = -\nabla \Phi$

$\nabla \cdot (\Phi \nabla \Phi) = |\nabla \Phi|^2 + \Phi \nabla^2 \Phi$

and $\int \nabla \cdot (\Phi \nabla \Phi) \longrightarrow 0$ if charge is localized

W is exclusively expressed in terms of $\vec{E}$ and no explicit reference to charges is used.

$$w = \frac{\epsilon_0}{2}|\vec{E}(\vec{x})|^2 \qquad (1.55)$$

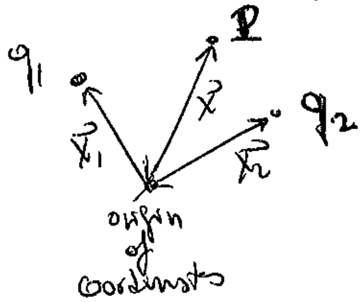
energy density

Makes sense: if the electric field is large, there has to be a lot of energy!

Note that w is ≥ 0, while the usual formula for the energy between, say, two charges is

$$\frac{q_i q_j}{4\pi\epsilon_0|\vec{x}_i-\vec{x}_j|}$$ which can be negative if one of the charges is negative.

To clarify this consider 2 points charges at $\vec{x}_1$ and $\vec{x}_2$:



At P, the electric field is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1(\vec{x}-\vec{x}_1)}{|\vec{x}-\vec{x}_1|^3} + \frac{q_2(\vec{x}-\vec{x}_2)}{|\vec{x}-\vec{x}_2|^3}\right)$$

Let us construct $\frac{\epsilon_0}{2}\vec{E}^2$:

$$\frac{\epsilon_0}{2}\vec{E}\cdot\vec{E} = \frac{(\epsilon_0/2)}{(4\pi\epsilon_0)^2}\left(\frac{q_1^2}{|\vec{x}-\vec{x}_1|^4} + \frac{q_2^2}{|\vec{x}-\vec{x}_2|^4} + \frac{2q_1q_2(\vec{x}-\vec{x}_1)\cdot(\vec{x}-\vec{x}_2)}{|\vec{x}-\vec{x}_1|^3|\vec{x}-\vec{x}_2|^3}\right)$$

These are "self-energy" terms namely they depend on the individual properties of each charge

The third term integrated over space is

$$\frac{1}{16\pi^2\epsilon_0} q_1 q_2 \int d^3x \; \frac{(\vec{x}-\vec{x}_1)\cdot(\vec{x}-\vec{x}_2)}{|\vec{x}-\vec{x}_1|^3\,|\vec{x}-\vec{x}_2|^3}$$

It can be shown (note: no need for the students to do it)

that this is

$\dfrac{q_1 q_2}{4\pi\epsilon_0}$ which is the energy "we thought" was the correct one.

So one formula has the self-energy and others not, so we have to be careful! This also affects the calculations of forces since they are obtained as $\dfrac{\Delta W}{\Delta x}$ for small displacements

Basically, in $\sum$ we knew "i" had to be $\neq$ "j", but when we move to the continuum $\int d^3x \int d^3x'$ we "lost" that constraint, causing the self-energy to appear in E

We skip Sections 1.12 and 1.13

Discussion about capacitance on page 43 left as reading material to students