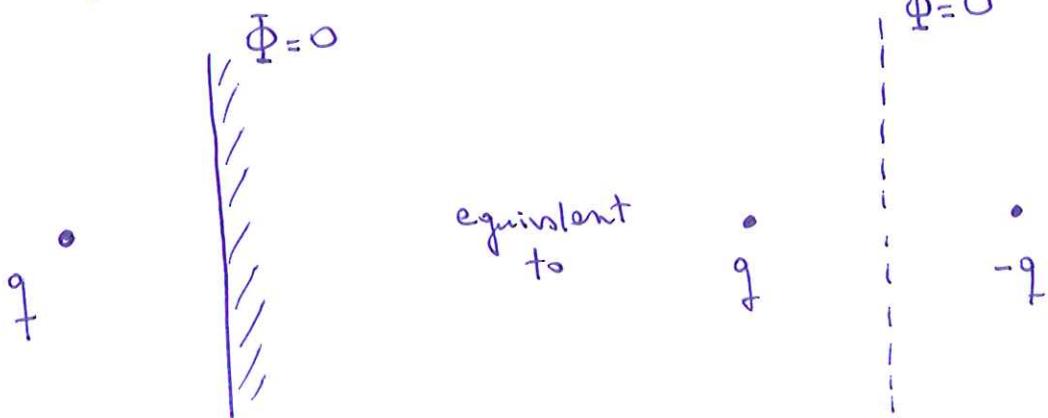


2.1 Method of images

Works when we have 1 or more point charges in the presence of boundary surfaces (such as conductors at fixed potential)

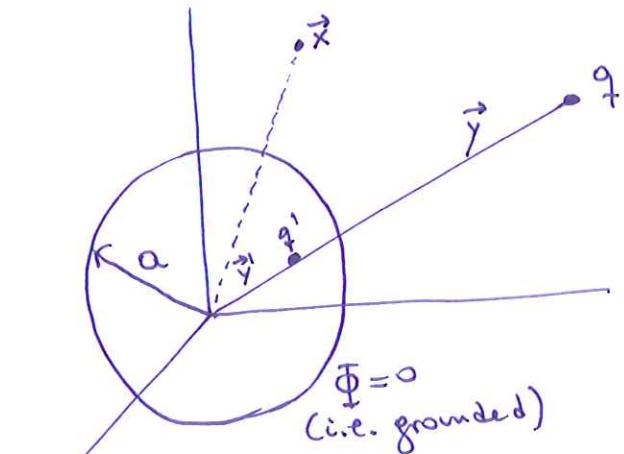
The main idea is that from the geometry of the problem it is possible to deduce that a small number of suitably placed charges, outside the region of interest, can simulate the boundary conditions. These extra charges are called "image charges" and the method is "method of images".

Example:



since we know two charges of opposite sign located at the same distance from the plane will give a $\Phi = 0$ at the plane.

2.2 Point charge in the presence of a conducting sphere



If there is an image charge that can replace the sphere by symmetry it will be located along the ray from the origin to the true charge.

$$\Phi(\vec{x}) = \frac{q/4\pi\epsilon_0}{|\vec{x} - \vec{y}|} + \frac{q'/4\pi\epsilon_0}{|\vec{x} - \vec{y}'|}$$

We must select q' and \vec{y}' such that $\Phi(|\vec{x}|=a) = 0$.

Consider a unit vector \vec{m} along the direction \vec{x} and \vec{m}' along

$$\Phi(\vec{x}) = \frac{q/4\pi\epsilon_0}{|\vec{x}\vec{m} - \vec{y}\vec{m}'|} + \frac{q'/4\pi\epsilon_0}{|\vec{x}\vec{m} - \vec{y}'\vec{m}'|} = \frac{q/4\pi\epsilon_0}{x|\vec{m} - \frac{\vec{y}\vec{m}'}{x}|} + \frac{q'/4\pi\epsilon_0}{y'|\vec{m}' - \frac{\vec{x}\vec{m}'}{y'}|}$$

$$\Phi(|\vec{x}|=a) = \frac{q/4\pi\epsilon_0}{a|\vec{m} - \frac{\vec{y}\vec{m}'}{a}|} + \frac{q'/4\pi\epsilon_0}{y'|\vec{m}' - \frac{\vec{x}\vec{m}'}{y'}|} \quad (\text{we do not know } q', y')$$

If we choose $\frac{q}{a} = -\frac{q'}{y'}$ and $\frac{y}{a} = \frac{a}{y'}$, then

$$\Phi(|\vec{x}|=a) = \frac{q/4\pi\epsilon_0}{a} \left[\frac{1}{\sqrt{(\vec{m} - \frac{\vec{y}\vec{m}'}{a}) \cdot (\vec{m} - \frac{\vec{y}\vec{m}'}{a})}} - \frac{1}{\sqrt{(\vec{m}' - \frac{\vec{x}\vec{m}'}{a}) \cdot (\vec{m}' - \frac{\vec{x}\vec{m}'}{a})}} \right] = 0$$

$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$

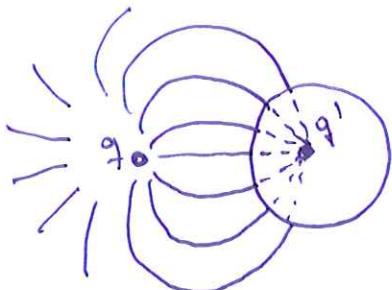
$$1 - \frac{2y}{a} \vec{m} \cdot \vec{m}' + \frac{y^2}{a^2}$$

$$1 - \frac{2y}{a} \vec{m} \cdot \vec{m}' + \frac{y^2}{a^2}$$

Thus,

$$q' = -\frac{a}{y} q \quad \text{and} \quad y' = \frac{a^2}{y}$$

is what we want



Let us now calculate something we did not know such as the charge density at the surface of the sphere.

We use $\mathbf{G} = \epsilon_0 (-\nabla \Phi)$

at $|\vec{x}|=a$
and normal
to the sphere.

We see at the back of book that

$$\nabla \Phi = e_1 \frac{\partial \Phi}{\partial r} + e_2 \frac{1}{r} \frac{\partial \Phi}{\partial \theta} + e_3 \frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi}$$

We only care about " $e_1 \frac{\partial \Phi}{\partial r}$ " and r is our x in the case we study.

Then

$$\mathbf{G} = -\epsilon_0 \left. \frac{\partial \Phi}{\partial x} \right|_{x=a}$$

$$\Phi = \frac{q/4\pi\epsilon_0}{|\vec{x} - \vec{y'}|} + \frac{q'/4\pi\epsilon_0}{|\vec{x} - \vec{y''}|}$$

$$\begin{aligned}
 & \textcircled{1} \quad \frac{\partial}{\partial x} \frac{q/4\pi\epsilon_0}{|\vec{x}\vec{m} - \vec{y}\vec{m}'|} = \frac{\partial}{\partial x} \frac{q/4\pi\epsilon_0}{\sqrt{x^2 - 2xy\cos\gamma + y'^2}} = \\
 & = q/4\pi\epsilon_0 \left(-\frac{1}{2}\right) \frac{1}{(x^2 - 2xy\cos\gamma + y'^2)^{3/2}} (2x - 2y\cos\gamma) \\
 & \stackrel{x=\alpha}{=} q/4\pi\epsilon_0 \left(-\frac{1}{2}\right) \frac{1}{(\alpha^2 - 2\alpha y\cos\gamma + y'^2)^{3/2}} (2\alpha - 2y\cos\gamma) \\
 & = \left(-\frac{1}{2}\right) \frac{q}{4\pi\epsilon_0} \frac{1}{y^3} \frac{1}{\left(1 + \frac{\alpha^2}{y^2} - \frac{2\alpha}{y}\cos\gamma\right)^{3/2}} (2\alpha - 2y\cos\gamma)
 \end{aligned}$$

$$\begin{aligned}
 & \textcircled{2} \quad \frac{\partial}{\partial x} \frac{q'/4\pi\epsilon_0}{|\vec{x}\vec{m} - \vec{y}'\vec{m}'|} = \frac{\partial}{\partial x} \frac{-\frac{\alpha}{y} q'/4\pi\epsilon_0}{\sqrt{x^2 - 2xy'\cos\gamma + y'^2}} = \\
 & = -\frac{\alpha}{y} q'/4\pi\epsilon_0 \left(-\frac{1}{2}\right) \frac{1}{(x^2 - 2xy'\cos\gamma + y'^2)^{3/2}} (2x - 2y'\cos\gamma) \\
 & \stackrel{x=\alpha, y'=\alpha/y}{=} -\frac{\alpha}{y} q'/4\pi\epsilon_0 \left(-\frac{1}{2}\right) \underbrace{\frac{1}{\left(\alpha^2 - 2\alpha\frac{\alpha^2}{y}\cos\gamma + \frac{\alpha^2}{y^2}\right)^{3/2}}}_{\alpha^3 \left(1 - 2\frac{\alpha}{y}\cos\gamma + \frac{\alpha^2}{y^2}\right)^{3/2}} (2\alpha - 2\frac{\alpha^2}{y}\cos\gamma)
 \end{aligned}$$

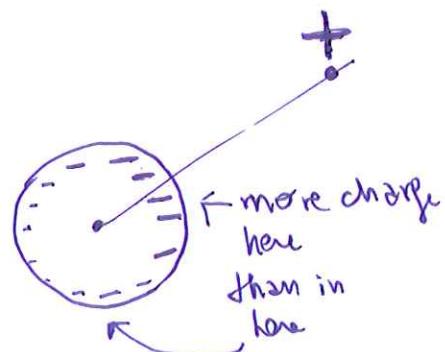
$$\textcircled{1} + \textcircled{2} = \frac{\frac{q}{4\pi\epsilon_0}}{\left(1 - \frac{2a}{y}\cos\delta + \frac{a^2}{y^2}\right)^{3/2}} \left[-\frac{1}{2y^3} (2a - 2y\cos\delta) + \frac{a}{2ya^3} (2a - 2\frac{a^2}{y}\cos\delta) \right]$$

$$-\frac{a}{y^3} + \frac{1}{y^2}\cos\delta + \frac{1}{ya} + \frac{1}{y^2}\cos\delta \\ = \frac{1}{ay} \left(1 - \frac{a^2}{y^2}\right)$$

$$\overbrace{-\epsilon_0[\textcircled{1} + \textcircled{2}]}^{\sigma} = \boxed{-\frac{q}{4\pi ay} \frac{1 - \frac{a^2}{y^2}}{\left(1 + \frac{a^2}{y^2} - \frac{2a}{y}\cos\delta\right)^{3/2}}} = \sigma$$

which is (2.5).

since $y > a$, then $1 - \frac{a^2}{y^2}$ is > 0 . Thus, σ is of sign opposite to q , as expected. For $\delta = 0$, i.e. $\vec{m} = \vec{n}^\perp$, then the denominator is minimized and σ is maximized. For $\vec{m} = -\vec{n}^\perp$ (or $\delta = \pi$), the opposite happens.



The integral of σ over the sphere gives q' (not shown).

Other things that we can calculate are for example the force at q' due to the sphere. This cannot be just the force caused by another fixed charge, because q' depends on the position of q , and as $y \rightarrow \infty$, $q' \rightarrow 0$. Let us do the math:

$$\begin{aligned}
 |\text{Force}| &= q' \frac{\partial}{\partial x} \frac{q'/4\pi\epsilon_0}{|\vec{x} - \vec{y}'|^3} = \frac{q'q'^*}{4\pi\epsilon_0} \frac{\partial}{\partial x} \frac{1}{\sqrt{x^2 - 2xy' + y'^2}} \Big|_{x=y} \\
 q' \cdot \vec{E} & \\
 &= \frac{q'q'^*}{4\pi\epsilon_0} \left(\frac{1}{2} \frac{2x - 2y'}{(x^2 - 2xy' + y'^2)^{3/2}} \right) = \frac{\frac{a^2 q^2}{y} \frac{2}{2} \left(y - \frac{a^2}{y} \right)}{2 \cdot 4\pi\epsilon_0 \underbrace{\left(y^2 - 2y \frac{a^2}{y} + \frac{a^4}{y^2} \right)^{3/2}}_{\left(y - \frac{a^2}{y} \right)^2}} \\
 &= \frac{\frac{a^2 q^2}{y} \frac{y \left(1 - \frac{a^2}{y^2} \right)}{\left(y - \frac{a^2}{y} \right)^3}}{\frac{1}{4\pi\epsilon_0} \frac{q^2 a^2 \left(1 - \frac{a^2}{y^2} \right)^{-2}}{y^3}} \\
 &= \boxed{\frac{1}{4\pi\epsilon_0} \cdot \frac{q^2}{a^2} \cdot \left(\frac{a}{y} \right)^3 \left(1 - \frac{a^2}{y^2} \right)^{-2}}
 \end{aligned}$$

which is (2.6)

Note that at y large then $\left(1 - \frac{a^2}{y^2} \right)^{-2} \approx 1$

and $\boxed{\text{Force} \sim \frac{1}{y^3}}$ as opposed to $\frac{1}{y^2}$ for a fixed charge. The reason is that q' decreases as y increases.

Note also that this force is attractive because the image charge has an opposite sign to q . Moreover, as $y \rightarrow a$, then $\frac{1}{1 - \frac{a^2}{y^2}} \rightarrow \infty$. Then, as $y \rightarrow a$ the density of charge $\sigma \rightarrow \infty$. This is not a problem or paradox because the integrated σ always is always q .

2.3 Point charge in the presence of a charged, insulated, conducting sphere.

not insulator, but a metal that is insulated

The idea here is that we add a charge Q to the metallic sphere and see what happens.

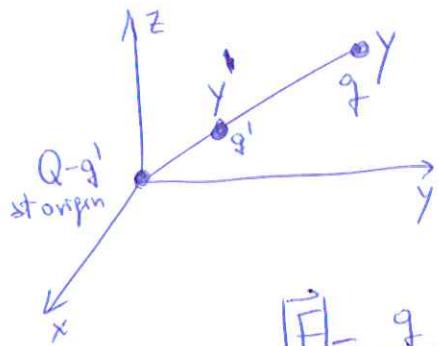
Jackson explains verbally an easy way to get the answer, i.e. Φ . He says that we can arrive to charge Q in 2 steps. First we start with the sphere grounded and in this case a charge q' will be generated as already explained. Then, we disconnect and add $Q - q'$ to the sphere. The key point is that the sphere with q' induced is in equilibrium with the external charge q . This induced charge q' could move but does not because the forces are balanced in the sphere. Namely \vec{E} tangential to sphere is 0 (normal to sphere is non-zero, but charge can't leave sphere).

Then, the extra $Q - q'$ will distribute itself uniformly!

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left[\underbrace{\frac{q}{|\vec{x}-\vec{y}|} - \frac{\frac{a}{r}q}{|\vec{x}-\frac{a^2}{r^2}\vec{y}|}}_{\text{From before}} + \underbrace{\frac{Q + \frac{a}{r}q}{|\vec{x}|}}_{\text{new}} \right].$$

$$\begin{aligned} \vec{y}' &= \vec{y}' \hat{n} = \vec{y}' \vec{y} \\ &= \frac{a^2}{r^2} \vec{y} \end{aligned}$$

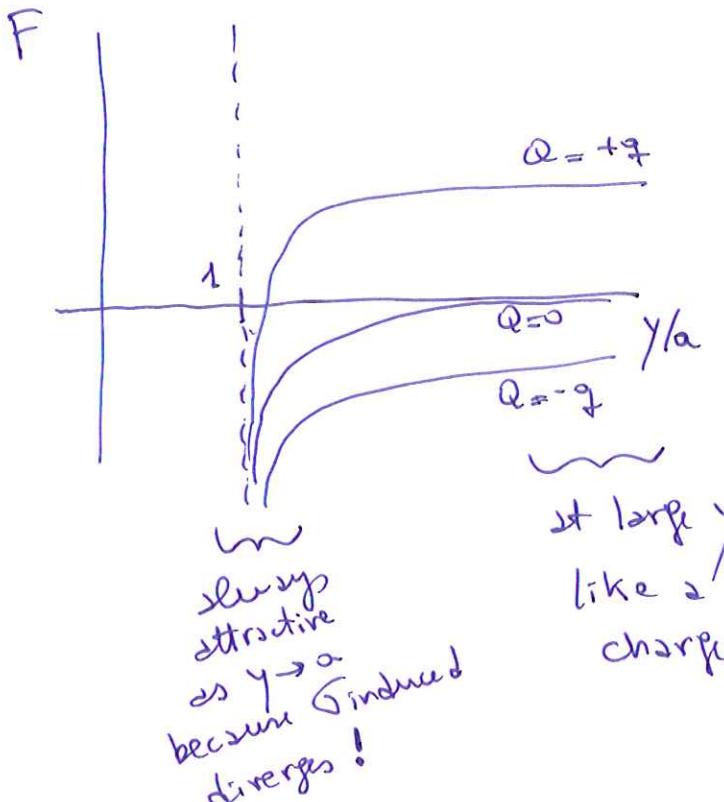
To calculate the force at g we simply think of the entire problem as made out of 3 charges:



Then, for the force at g we use Coulomb's forces caused by the other two:

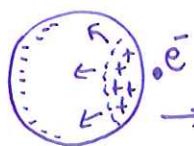
$$\begin{aligned}
 \vec{F} &= \frac{q}{4\pi\epsilon_0} \left[\frac{q'}{|g-g'|^2} + \frac{(Q-q')}{|g|^2} \right] = \\
 &= \frac{q}{4\pi\epsilon_0} \left[\frac{-\frac{a}{4}q}{\left(\sqrt{y^2 - y'^2}\right)^2} + \frac{\left(Q + \frac{a}{4}q\right)}{y^2} \right] = \frac{q}{4\pi\epsilon_0} \left[\frac{-\frac{a}{4}q}{y^2 \left(1 - \frac{a^2}{4y^2}\right)^2} + \frac{\left(Q + \frac{a}{4}q\right)}{y^2} \right] \\
 &\quad \downarrow \\
 &\quad \left(y - \frac{a^2}{4}\right) \cancel{\frac{\vec{m}}{m^2}} \cancel{\frac{1}{1}} \\
 &= \frac{q(Q)}{4\pi\epsilon_0 y^2} - \frac{q^2}{4\pi\epsilon_0} \left[\frac{\left(a/4\right)}{y^2 \left(1 - \frac{a^2}{4y^2}\right)^2} - \frac{\left(a/4\right)}{y^2} \right] \\
 &= \frac{q}{4\pi\epsilon_0 y^2} \left[Q - \frac{qa}{4} \left(\frac{1}{\left(1 - \frac{a^2}{4y^2}\right)^2} - 1 \right) \right] \\
 &\quad \downarrow \quad y^4 + a^4 - 2y^2 a^2 \\
 &\quad \frac{a}{4} \left[\frac{y^4 - (y^2 - a^2)^2}{(y^2 - a^2)^2} \right] = \frac{a(-a^4 + 2y^2 a^2)}{4 \left(y^2 - a^2 \right)^2} = \frac{a^3 (2y^2 - a^2)}{4 \left(y^2 - a^2 \right)^2} \\
 &\quad \text{which is } \underline{\underline{(2.9)}}.
 \end{aligned}$$

The force can be calculated Eq.(2.9) and it can be plotted and the result is



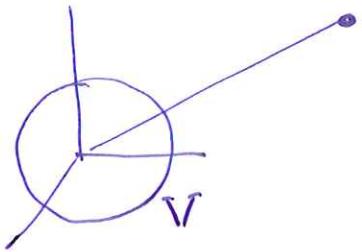
this attraction is present even for $Q=0$.

It explains the notion of the "work function of a metal". To remove from a metal a charge even of the same sign as the rest, work has to be done, because of the image charge that is of the opposite sign.



→ if you want to remove an e^- from close to the surface, it is attracted back by the induced charge.

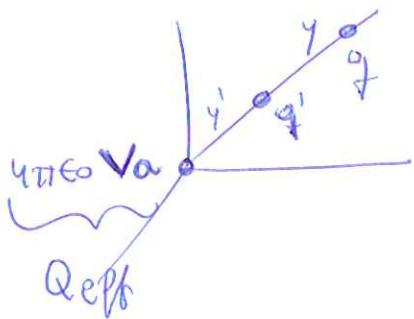
2.4 Point charge near a conducting sphere at fixed potential



Same reasoning as in 2.3, namely first we think of the sphere as connected to the ground, then it reaches equilibrium and we know from 2.2 that $\Phi(x=0) = 0$. We simply need to add "something" that gives a potential V and that is a pointlike charge at the center with $q = Va/4\pi\epsilon_0$.

$$\Phi(\vec{x}) = \underbrace{\frac{1}{4\pi\epsilon_0} \left[\frac{q}{|\vec{x}-\vec{y}|} - \frac{aq}{y|\vec{x}-\frac{a^2}{y}\vec{y}|} \right]}_{\text{as before}} + \underbrace{\frac{Va}{|\vec{x}|}}_{\text{new}}$$

To calculate the force again we remember that this is like 3 particles:

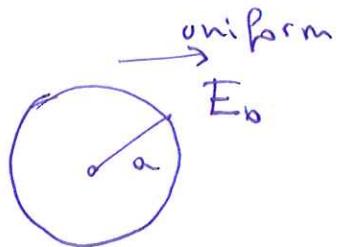


Then, it is sufficient to take Eq. (2.9) and replace Q by $4\pi\epsilon_0 Va + q'$

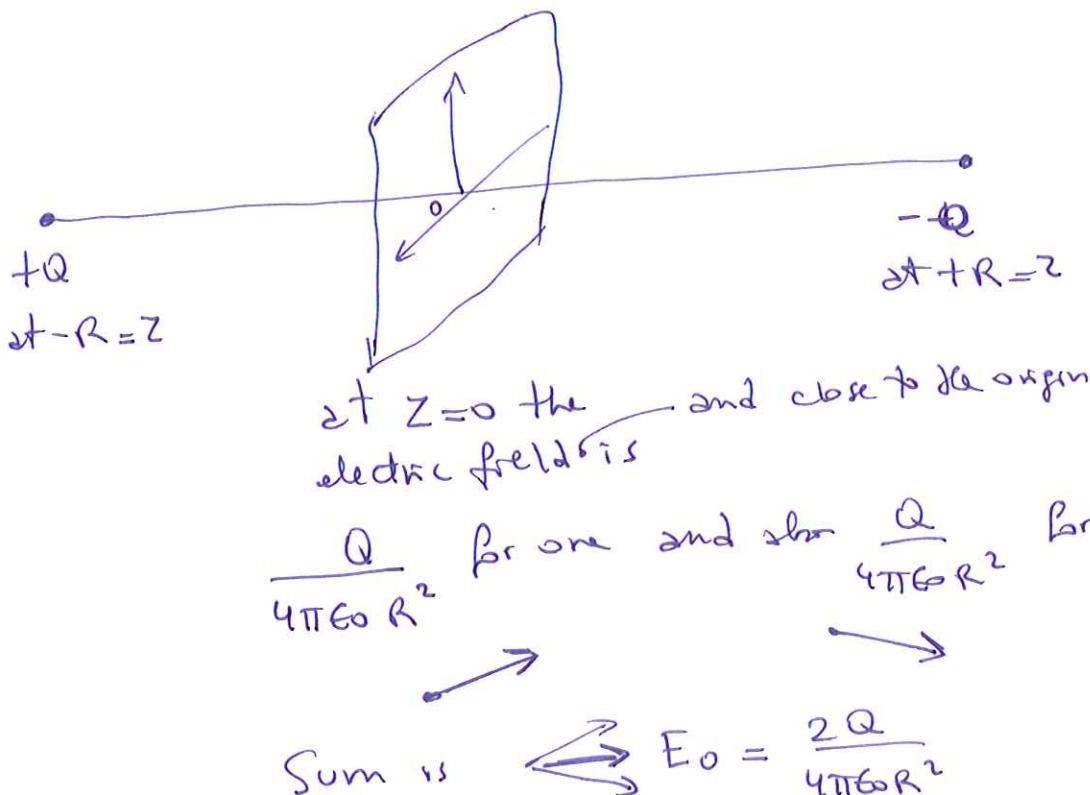
$$\begin{aligned}
 |\vec{F}| &= \frac{q}{4\pi\epsilon_0 y^2} \left[(4\pi\epsilon_0 Va + q') - \frac{q a^3 (2y^2 - a^2)}{4(y^2 - a^2)^2} \right] \\
 &= \frac{q}{y^2} \left[Va - \frac{1}{4\pi\epsilon_0} \underbrace{\left[\frac{+aq}{y} + \frac{q a^3}{4} \cdot \frac{(2y^2 - a^2)}{(y^2 - a^2)^2} \right]}_{aq} \right] \\
 &\quad aq \left[\frac{(y^2 - a^2)^2 + a^2 (2y^2 - a^2)}{4(y^2 - a^2)^2} \right] = \\
 &= \frac{aq}{(y^2 - a^2)^2 y} \left[y^4 + a^2 y^2 - 2y^2 a^2 + 2a^2 y^2 - a^4 \right] \\
 &= \frac{q a y^3}{(y^2 - a^2)^2}
 \end{aligned}$$

which is (2.11)

2.5 Conducting Sphere in a Uniform Electric Field by Method of Images



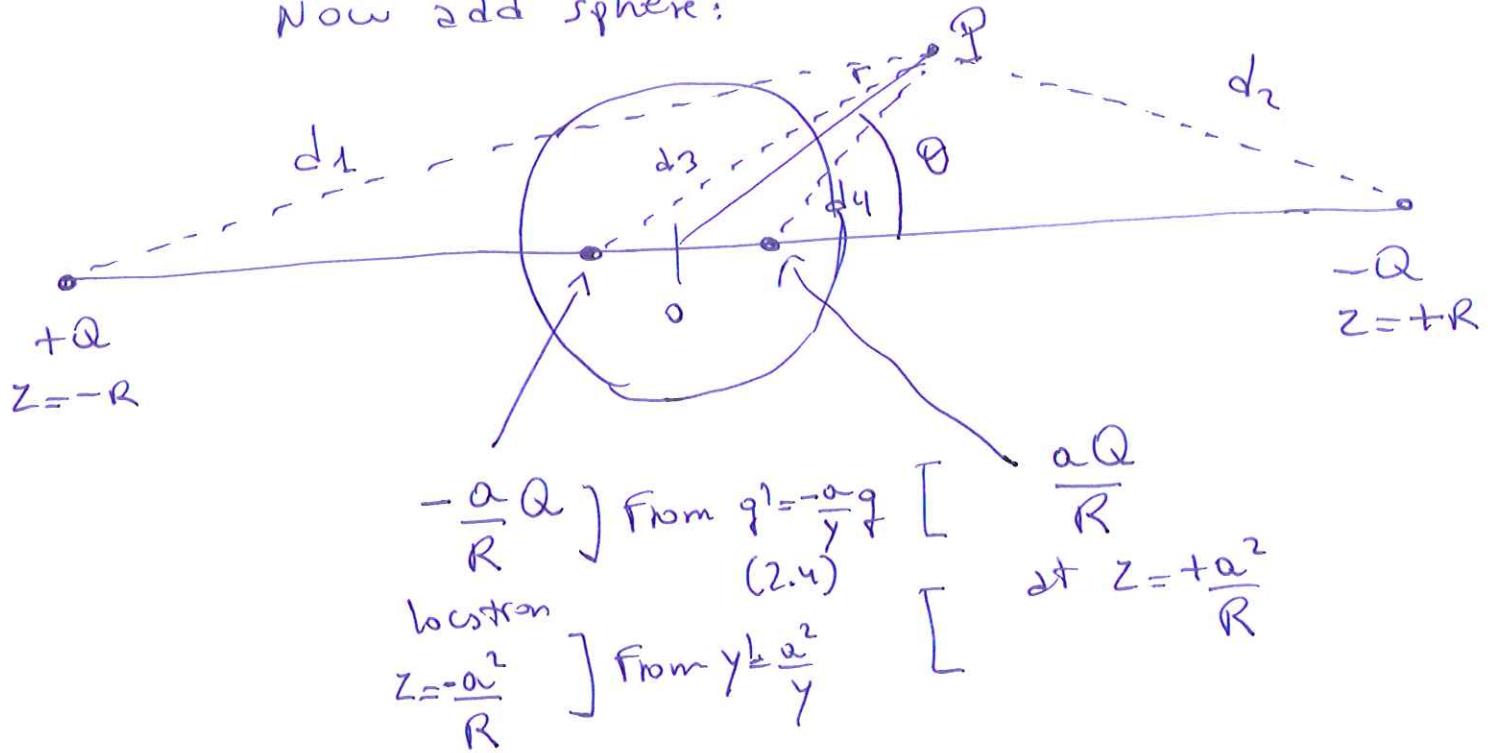
Trick: consider 2 charges $\pm Q$ at positions $\mp R$:



So in a region of dimensions much smaller than R near the origin, the electric field is $\frac{2Q}{4\pi\epsilon_0 R^2}$ and points along z -axis.

As $R, Q \rightarrow \infty$ such that $\frac{Q}{R^2} = \text{constant}$
this is exact.

Now add sphere:



Then, Φ will have 4 components:

$$\Phi = \frac{Q}{4\pi\epsilon_0} \cdot \frac{1}{\sqrt{r^2 + R^2 - 2rR \cos(\theta) - \cos^2\theta}} \leftarrow d_1$$

$$\frac{-Q}{4\pi\epsilon_0} \cdot \frac{1}{\sqrt{r^2 + R^2 - 2rR \cos\theta}} \leftarrow d_2$$

$$\frac{-\alpha Q}{R 4\pi\epsilon_0} \cdot \frac{1}{\sqrt{r^2 + \frac{\alpha^2}{R^2} - 2r \frac{\alpha^2}{R} \cos(\theta) - \cos^2\theta}} \leftarrow d_3$$

$$\frac{\alpha Q}{R 4\pi\epsilon_0} \cdot \frac{1}{\sqrt{r^2 + \frac{\alpha^2}{R^2} - 2r \frac{\alpha^2}{R} \cos\theta}} \leftarrow d_4$$

Now we have to Taylor expand:

$$\frac{1}{\sqrt{r^2 + R^2 + 2rR \cos\theta}} = \frac{1}{R} \cdot \frac{1}{\sqrt{1 + \frac{r^2}{R^2} + \frac{2r}{R} \cos\theta}}$$

$\approx 1 - \frac{1}{2} \left(\frac{2r}{R} \cos\theta \right)$

$$\frac{1}{\sqrt{r^2 + R^2 + 2rR \cos\theta}} \approx \frac{1}{R} \left(1 + \frac{1}{2} \frac{2r}{R} \cos\theta \right)$$

all together

$$\begin{aligned} & \approx \frac{Q}{4\pi\epsilon_0} \frac{1}{R} \left(1 - \frac{r}{R} \cos\theta \right) - \frac{Q}{4\pi\epsilon_0} \frac{1}{R} \left(1 + \frac{r}{R} \cos\theta \right) \\ & = \boxed{-\frac{2Q}{4\pi\epsilon_0} \frac{r}{R^2} \cos\theta} \quad (\text{first term in (2.13)}) \end{aligned}$$

$$\begin{aligned} \frac{1}{\sqrt{r^2 + \frac{a^2}{R^2} + \frac{2a^2 r}{R} \cos\theta}} &= \frac{1}{r} \frac{1}{\sqrt{1 + \frac{a^4}{R^2 r^2} + \frac{2a^2}{Rr} \cos\theta}} \\ &\approx \frac{1}{r} \left(1 \mp \frac{1}{2} \frac{2a^2}{Rr} \cos\theta \right) \end{aligned}$$

Then the last two terms of (2.12) become:

$$\begin{aligned} & -\frac{aQ/4\pi\epsilon_0}{Rr} \left(1 - \frac{a^2}{Rr} \cos\theta \right) \rightarrow \frac{2aQ}{4\pi\epsilon_0 Rr} \frac{a^2}{Rr} \cos\theta \\ & + \frac{aQ/4\pi\epsilon_0}{Rr} \left(1 + \frac{a^2}{Rr} \cos\theta \right) \rightarrow \frac{2aQ}{4\pi\epsilon_0 Rr} \frac{a^3}{R^2 r^2} \cos\theta \\ & = \boxed{\frac{1}{4\pi\epsilon_0} \frac{2a^3 \cos\theta}{R^2 r^2}} \end{aligned}$$

second term in (2.11)

Thus, $\Phi \approx \frac{1}{R \gg a \{4\pi\epsilon_0\} r} \left[-\frac{2Qr}{R^2} \cos\theta + \frac{2Qa^3}{R^2 r^2} \cos\theta \right]$

Since $E_0 = \frac{2Q}{4\pi\epsilon_0 R^2}$, then

$$\begin{aligned}\Phi &= -E_0 \left(r \cos\theta - \frac{a^3}{r^2} \cos\theta \right) \\ &= \boxed{-E_0 \left(r - \frac{a^3}{r^2} \right) \cos\theta}\end{aligned}$$

The first term is the potential of a uniform electric field. The second term comes from the induced surface charge density (or, the image charges).