

3.1 Laplace Equation in Spherical Coordinates

) In spherical coordinates, from the back cover of the book, we know that:

$$\nabla^2 \Phi = \frac{1}{r} \frac{\partial^2 (r\Phi)}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

Like we did in separation of variables we assume:

$$\Phi = \underbrace{\frac{U(r)}{r}} P(\theta) Q(\phi)$$

Instead of $F(r)$, we use $\frac{U(r)}{r}$ for convenience.

) Then,

$$\nabla^2 \Phi = PQ \frac{1}{r} \frac{\partial^2 (rU)}{\partial r^2} + \frac{UQ}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial P}{\partial \theta} \right) + \frac{UP}{r^2 \sin^2 \theta} \frac{\partial^2 Q}{\partial \phi^2} = 0$$

or, dropping a common $1/r$, we get:

$$PQ \cdot \frac{d^2 U}{dr^2} + \frac{UQ}{r^2 \sin \theta} \cdot \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) + \frac{UP}{r^2 \sin^2 \theta} \cdot \frac{d^2 Q}{d\phi^2} = 0$$

Multiplying by $\frac{r^2 \sin^2 \theta}{UPQ}$, we get:

$$r^2 \sin^2 \theta \left[\frac{1}{U} \frac{d^2 U}{dr^2} + \frac{1}{P r^2 \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) \right] + \frac{1}{Q} \frac{d^2 Q}{d\phi^2} = 0$$

All the " ϕ " dependence is in the last term.

Thus,
$$\frac{1}{Q} \frac{d^2 Q}{d\phi^2} = -m^2 \text{ (constant unknown)}$$

with solutions
$$Q = e^{\pm im\phi}$$

m has to be an integer for ϕ to be single value.

Then, we get for the rest:

$$\frac{1}{U} \frac{d^2 U}{dr^2} + \frac{1}{r^2 \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dP}{d\theta} \right) - \frac{m^2}{r^2 \sin^2\theta} = 0$$

Multiplying by r^2 we get:

$$\underbrace{\frac{r^2}{U} \frac{d^2 U}{dr^2}}_{\text{all } r \text{ dependence}} + \underbrace{\frac{1}{P \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dP}{d\theta} \right) - \frac{m^2}{\sin^2\theta}}_{\text{all } \theta \text{ dependence}} = 0$$

they must be equal to a constant, each with different sign.

We call this constant " $l(l+1)$ " and temporarily it is "real"

$$\frac{r^2}{U} \frac{d^2 U}{dr^2} = l(l+1) \rightarrow \boxed{\frac{d^2 U}{dr^2} - \frac{l(l+1)}{r^2} U = 0}$$

and

$$\frac{1}{P \sin \theta} \cdot \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = -l(l+1)$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) + \left[l(l+1) - \frac{m^2}{\sin^2 \theta} \right] P = 0$$

For the radial one we try

$$U = A r^{l+1} + B r^{-l}$$

$$\left[\text{then, } \frac{dU}{dr} = A(l+1)r^l + B(-l)r^{-l-1} \right.$$

$$\frac{d^2U}{dr^2} = A(l+1)l r^{l-1} + B(-l)(-l-1)r^{-l-2}$$

$$= l(l+1) \underbrace{\left[A r^{l-1} + B r^{-l-2} \right]}_{\text{proportional } U/r^2} = \frac{l(l+1)U}{r^2} \quad \checkmark \quad \left. \right]$$

Section 3.2

The equation for P is a bit more complicated.

Let us change variables to $x = \cos \theta$, $\sin \theta = \sqrt{1-x^2}$.

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) = \frac{1}{\cancel{\sin \theta}} \underbrace{\frac{d \cos \theta}{d\theta}}_{-\cancel{\sin \theta}} \frac{d}{d \cos \theta} \left(\sin \theta \underbrace{\frac{d \cos \theta}{d\theta} \frac{dP}{d \cos \theta}}_{-\cancel{\sin \theta}} \right)$$

$$= - \frac{d}{dx} \left[-(1-x^2) \frac{dP}{dx} \right] = \frac{d}{dx} \left[(1-x^2) \frac{dP}{dx} \right]$$

Thus:

$$\frac{d}{dx} \left[(1-x^2) \frac{dP}{dx} \right] + \left[l(l+1) - \frac{m^2}{1-x^2} \right] P = 0$$

eg. "Generalized Legendre equation"
solutions "associated Legendre functions".

We will not deal with the math of pages 97-101 of the book when dealing with the "Legendre polynomials" since this is learned elsewhere. We will just outline the results:

For $m=0$:

1. $l=0$ or a positive integer, otherwise the

solution is "ill-behaved" \rightarrow divergences at $x = \pm 1$ if

2. P is a Legendre polynomial

P is a series instead of a polynomial.

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

.....

Just plug into (3.10) and check a few!

$$\left[\begin{array}{l} \text{Ex: } l=2, \frac{dP}{dx} = 3x, (1-x^2)\frac{dP}{dx} = 3x - 3x^3, \frac{d}{dx}(3x - 3x^3) = 3 - 9x^2 \\ l(l+1)P = 2 \cdot 3 \cdot \frac{1}{2}(3x^2 - 1) = 9x^2 - 3 \end{array} \right] \checkmark$$

$$3. \int_{-1}^1 P_{\ell}^1(x) P_{\ell}(x) dx = \frac{2}{2\ell+1} \delta_{\ell\ell'} \quad (3.21)$$

The exercise at the end of page 99. (step function)
is interesting and should be
done by the students.

3.3 Problems with Azimuthal Symmetry

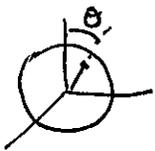
) In this case, we need the result to be independent of ϕ , thus basically it is $m=0$.

In this case:

$$\Phi = \sum_{m=0}^{\infty} \frac{U(r)}{r} P(\theta) = \sum_{l=0}^{\infty} \frac{A_l r^{l+1} + B_l r^{-l}}{r} P_l(\theta) ; \quad A_l, B_l \text{ are determined from boundary conditions.}$$

the most general solution is a sum over l

Example: $\Phi = V(\theta)$ on the surface of a sphere of radius a (it does not depend on ϕ , thus it is azimuthal)
Find Φ inside the sphere.



Then:
$$V(\theta) = \sum_{l=0}^{\infty} (A_l a^l + B_l a^{-l-1}) P_l(\theta)$$

If there are no charges at the origin, as it occurs in this example, then Φ should be finite at $r=0$. Then, $B_l \equiv 0$ otherwise r^{-l-1} diverges as $r \rightarrow 0$. Then:

$$V(\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos \theta)$$

$$\sin \theta P_m(\cos \theta) V(\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos \theta) P_m(\cos \theta) \sin \theta$$

$$\int_0^{\pi} \sin \theta P_m(\cos \theta) V(\theta) d\theta = \sum_{l=0}^{\infty} A_l a^l \underbrace{\int_0^{\pi} \sin \theta d\theta P_l(\cos \theta) P_m(\cos \theta)}_{\cos \theta = x}$$

$$-\sin \theta d\theta = dx ; \quad \int_0^{\pi} \rightarrow \int_1^{-1} = -\int_{-1}^1$$

Then, the integral on the right becomes

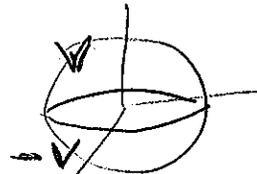
$$\int_{-1}^1 dx P_l(x) P_m(x) \text{ which is } \frac{2}{2l+1} \delta_{lm} \quad \text{Eq. (3.21)}$$

$$\int_0^\pi d\theta \sin\theta P_m(\cos\theta) V(\theta) = A_m a^m \frac{2}{2m+1}$$

or going back to l :

$$A_l = \frac{2l+1}{2a^l} \int_0^\pi V(\theta) P_l(\cos\theta) \sin\theta d\theta \quad (3.35)$$

If the potential $V(\theta)$ is



as we did before in the Green's function context, it can be shown (page 101) that the result obtained by this method is the same.

Skip pages 102-103

Skip Section 3.4

3.5 Associated Legendre Functions and Spherical Harmonics

Let us consider general potential problems with $m \neq 0$, i.e. without azimuthal symmetry. Thus, we must solve Eq. (3.6) for generic "l" and "m".

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) + \left[l(l+1) - \frac{m^2}{\sin^2 \theta} \right] P = 0$$

or Eq. (3.9) after $x = \cos \theta$.

It can be shown that there are solutions that are finite on the interval $-1 \leq x \leq 1$ when

$$l = 0, 1, 2, 3, \dots$$

$$m = -l, -(l-1), \dots, 0, \dots, (l-1), l$$

P_l^m are called "associated Legendre functions"

Combining the " P_l^m " solution of (3.6) or (3.9) with $e^{im\phi}$ we define the "spherical harmonics"

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \underbrace{P_l^m(\cos \theta)}_{\text{see (3.49)}} e^{im\phi}$$

where

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \ Y_{l_1 m_1}^*(\theta, \phi) Y_{l_2 m_2}(\theta, \phi) = \delta_{l_1 l_2} \delta_{m_1 m_2}$$

↑
orthonormal functions

Special cases:

$$l=0 \quad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$l=1 \quad \begin{cases} Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \\ Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \end{cases}$$

$$l=2 \quad \begin{cases} Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2\theta e^{2i\phi} \\ Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi} \\ Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2\theta - \frac{1}{2} \right) \end{cases}$$

.....

The general solution we are after of the potential problem now is:

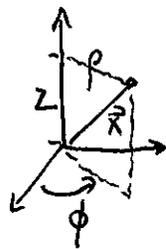
$$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left[A_{lm} r^l + B_{lm} r^{-(l+1)} \right] Y_{lm}(\theta, \phi)$$

From $\frac{U(r)}{r}$

From $P(\theta)Q(\phi)$

3.7 Laplace equation in cylindrical coordinates; Bessel functions

From back of the book, in cylindrical coordinates the Laplace equation becomes:



$$\frac{d^2\Phi}{d\rho^2} + \frac{1}{\rho} \frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2} \frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

The math is similar to that employed for a problem with spherical coordinates:

1. Propose $\Phi = R(\rho)Q(\phi)Z(z)$

2. Replace and find 3 diff. eqs., one for each coordinate

3. $Z(z) = e^{\pm kz}$
 $Q(\phi) = e^{\pm i\nu\phi}$

$\frac{\partial^2\Phi}{\partial z^2}$ term $\rightarrow \frac{\partial^2 Z}{Z \partial z^2} = +k^2$

Like before, for two of them we can find generic solutions.

positive. Note that in homework problem 3.9 we need to take $+k$ instead of $+k$

Like before, instead of "l" "m" we have now "k" and "nu". nu is found to be an integer while k is real and positive

4. At the end, we have one complicated equation to solve.

The solutions of that remaining equation are the Bessel functions that are described in detail in the book from page 112 to 116. We will not get into these details, but simply use the Bessel functions.

A bit more detail:

$$\frac{d^2 R}{dp^2} + \frac{1}{p} \frac{dR}{dp} + \left(k^2 - \frac{\nu^2}{p^2} \right) R = 0$$

$$x = kp$$

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2} \right) R = 0 \quad (\text{Bessel Equation})$$

The solutions are called "Bessel functions of order ν "

Solutions are Bessel functions $J_\nu(x)$

and Neumann functions $N_\nu(x)$

that are given in the book as a series in Eqs. (3.82) and (3.83)

Thus, a generic solution of Φ involves a linear combination of J_ν and N_ν

(remember ν is an integer)

Note that J_ν has a normalization condition (3.95) that can be used to find unknown coefficients.

Detail:

In equation (3.73) $\frac{d^2 Z}{dz^2} - k^2 Z = 0$

I could have selected " $-k^2$ " instead of " k^2 ".

For instance, in a problem involving a long (∞) cylinder I may need $e^{\pm kz}$ as solution, but in a finite cylinder with $\Phi = 0$, at say $z = 0$ and L , I would prefer sine and cosine solutions.

If I use the opposite sign as before then in the Bessel equation I have to change $k^2 \rightarrow -k^2$ and I get

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) R = 0$$

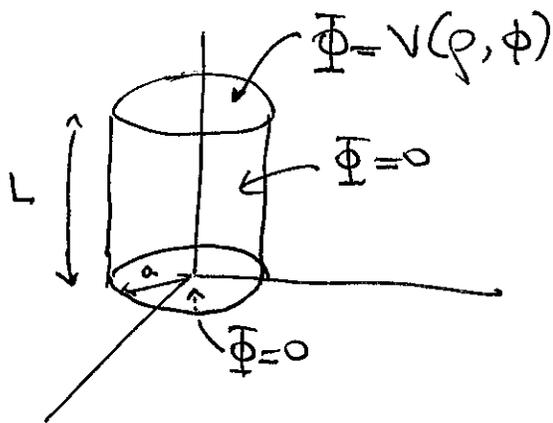
The solutions are called "modified Bessel functions"

and they are written as

| | | |
|---|-----|---|
| $I_\nu(x)$ | and | $K_\nu(x)$ |
|  | |  |
| $\rightarrow 0$ | | $\rightarrow \infty$ |
| as $x \rightarrow 0$ | | as $x \rightarrow 0$ |

3.8 Boundary problems in Cylindrical Coordinates

Consider a problem as shown in the figure:



Find Φ at any point inside the cylinder.

Like we did in other cases before we will ask the potential to be single valued.

This makes " v " to be an integer m and thus a generic Q will be:

$$Q(\phi) = A \sin(m\phi) + B \cos(m\phi)$$

(If there is any doubt simply recall the original equation

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \quad (3.74) \text{ and check that the proposed solution is a solution.)}$$

$Z(z)$ could be a linear combination of $e^{\pm kz}$

However, $\Phi = 0$ at $z=0$ (it is one of the boundary conditions)

thus if we propose a generic $C e^{kz} + D e^{-kz}$ that will not happen. But selecting $D = -C$ then we have this OK.

Thus, the general form for this problem is:

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh(k_{mn}z) \cdot [A_{mn} \sin(m\phi) + B_{mn} \cos(m\phi)]$$

It satisfies $\Phi = 0$ at $z=0$ and at $\rho=a$ and it is finite at $\rho=0$, and it is single valued.

We still need ^{to use} the condition $\Phi = V(\rho, \phi)$ at $z=L$.

$$\underbrace{V(\rho, \phi)}_{\text{given}} = \sum_{m,n} J_m(k_{mn}\rho) \sinh(k_{mn}L) (A_{mn} \sin m\phi + B_{mn} \cos m\phi)$$

The coefficients A_{mn} and B_{mn} can be obtained by the "usual" procedure involving orthogonality of functions, etc.

In particular we have to use Eqs (2.36, 2.37) since in ϕ we are dealing with a Fourier series. But we also need Eq. (3.95) since we are dealing with a Bessel series.

The results for A_{mn} and B_{mn} are in Eqs. (3.105b) and they will not be reproduced here.

↑
i.e. I must $\int d\phi$ after multiplying by "sin" or "cos" and
I must $\int \rho d\rho$ after multiplying by "J_m", as usual.