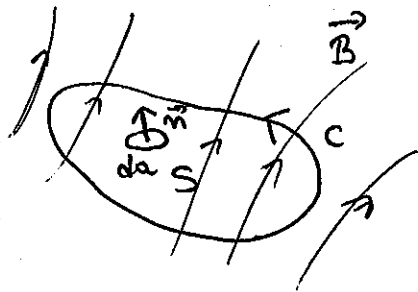# 5.15 Faraday's law of induction

Faraday did experiments involving a circuit in a time-dependent magnetic field. He observed that in the circuit a current was induced when:

(1) the current in another nearby circuit was turned on and off

(2) the second circuit with a current was moved relative to the first

(3) a permanent magnet was moved in and out of the circuit

The "magnetic flux" is defined as:

$$F = \int_S \vec{B} \cdot \vec{n} \, da$$

The "electromotive force" is defined as

$$\mathcal{E} = \oint_C \vec{E}' \cdot \vec{dl}$$

↑ electric field at $\vec{dl}$.

Faraday's experiments led to:

$$\mathcal{E} = -k \frac{dF}{dt}$$

the induced current
opposes the change of flux

Example:

If B increases with time
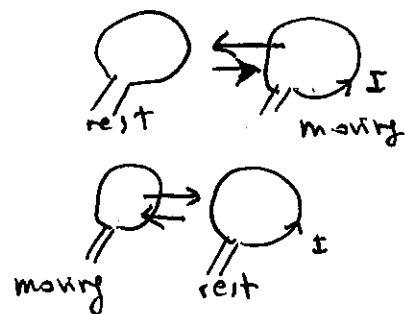then $\frac{dF}{dt} > 0$. Thus, the induced
current is circulating so

because it generates a B pointing
so shown

) In our unit system (SI), $K = 1$.

$\left(\vec{E}' \text{ is the electric field at } \vec{dl} \text{ in its "rest frame of coordinates"} \right.$

↓ the very nice discussion about
galilean transformations is left
to the students to read. It
addresses the fact that in the
previous page, example (2), which
of the two circuits is moving
should be irrelevant.

Let us write Faraday's law in differential form.

$$\oint_C \vec{E}' \cdot d\ell = \int_S (\nabla \times \vec{E}) \cdot \vec{n} \, da = -\frac{d}{dt} \int_S \vec{B} \cdot \vec{n} \, da = \int_S \left(-\frac{\partial \vec{B}}{\partial t}\right) \cdot \vec{n} \, da$$

↑ Stoke's theorem

↑ If circuit is held **fixed** in the chosen reference frame i.e. "S" and "C" are not changing.
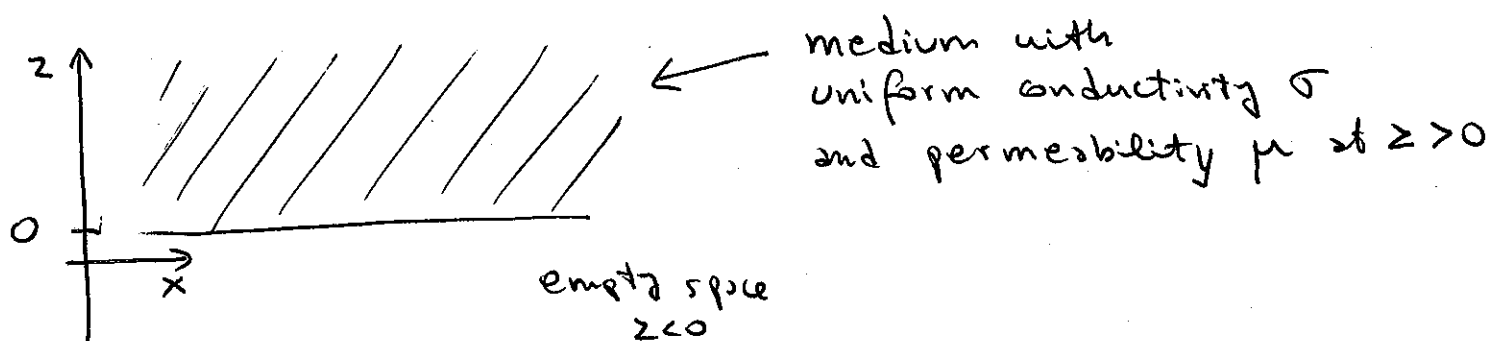
Thus:

$$\boxed{\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0}$$

Generalization of $\nabla \times \vec{E} = 0$ to time dependent fields.

We leave Sections 5.16 and 5.17 for the students to handle.

5.18 A. Skin depth, Eddy currents, ...

medium with uniform conductivity σ and permeability μ at $z > 0$

empty space $z < 0$

At the surface we introduce a magnetic field in the x direction

$$H_x(t) = H_0 \cos(\omega t)$$

that is not changing in space but changes in time.

What happens inside the medium?

From $\nabla \times \vec{H} = \vec{J}$, we can say that in the medium

$$\nabla \times \vec{B} = \mu \vec{J}$$ since μ in the medium is constant.

In addition, let us assume Ohm's law to be valid:

$$\vec{J} = \sigma \vec{E}.$$

Then: $$\nabla \times \vec{B} = \mu \sigma \vec{E}$$

$$\underbrace{\nabla \times (\nabla \times \vec{A})}_{} = \mu \sigma \vec{E} \stackrel{=}{=}$$

$$\underbrace{\nabla (\nabla \cdot \vec{A})}_{=0 \text{ in Coulomb's gauge}} - \nabla^2 \vec{A}$$

From Faraday's law:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{E} = \nabla \times \left(-\frac{\partial \vec{A}}{\partial t}\right)$$

$$= -\mu \sigma \frac{\partial \vec{A}}{\partial t}$$

Then:

$$\boxed{\nabla^2 \vec{A} = \mu\sigma \frac{\partial \vec{A}}{\partial t}}$$

(5.160)

↑

Note: In

$$\nabla \times \vec{E} = \nabla \times \left(-\frac{\partial \vec{A}}{\partial t}\right)$$

we could have said

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \Phi$$

To write $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$ we need

to assume also:

$$\left[\begin{array}{l} \text{negligible free charge} \\ \text{i.e. time varying } B \\ \underline{only} \text{ source of } \vec{E}. \end{array}\right]$$

What boundary conditions?

From (5.86) and (5.87), and knowing that there is no surface current density, then:

$$(\vec{B_2} - \vec{B_1}) \cdot \vec{n} = 0$$

$$\vec{n} \times (\vec{H_2} - \vec{H_1}) = 0$$

Since at $z < 0$ there is no B_z, then inside the medium the z component is also 0, from $\vec{B_2} \cdot \vec{n} = \vec{B_1} \cdot \vec{n}$

$$B_2^z = B_1^z.$$

From $\vec{n} \times \vec{H_2} = \vec{n} \times \vec{H_1}$:

$$\vec{H}_1 = (H_x, 0, 0)$$
$$\vec{n} = (0, 0, 1)$$

)

Then, $\vec{n} \times \vec{H}_1 = \begin{vmatrix} e_x & e_y & e_z \\ 0 & 0 & 1 \\ H_x & 0 & 0 \end{vmatrix} = H_x \, \hat{e}_y$ (outside medium, $z<0$)

$\left. \phantom{\frac{1}{1}} \right\} = $

$\vec{n} \times \vec{H}_2 = \begin{vmatrix} e_x & e_y & e_z \\ 0 & 0 & 1 \\ H_x^2 & H_y^2 & H_z^2 \end{vmatrix} = -H_y^2 \, \hat{e}_x + H_x^2 \, \hat{e}_y$ ($z>0$)

then: $H_y^2 = 0$ for $z > 0$ (actually $z = 0^+$)

$H_x^2 = H_x = H_0 \cos \omega t$ (at $z = 0^+$).

) Once we know that in the medium $z > 0$ the boundary condition is such that $\vec{H}$ at $z = 0^+$ points along x, then <u>$\vec{H}$ has only an x component</u> <u>in the entire $z > 0$ domain</u> because $\nabla^2 \vec{A} = \mu \sigma \dfrac{\partial \vec{A}}{\partial t}$

is linear. For instance, for the y-component

$$\nabla^2 A_y = \mu \sigma \frac{\partial A_y}{\partial t}. \quad \text{But at } z = 0^+, \left[\begin{array}{l} A_x = A_y = 0 \\ A_z = H_0 + (z,t) \end{array}\right]$$

$\nabla \times \vec{A} = \begin{vmatrix} e_x & e_y & e_z \\ \nabla_x & \nabla_y & \nabla_z \\ A_x & A_y & A_z \end{vmatrix} = $ since

$= \begin{vmatrix} e_x & e_y & e_z \\ \nabla_x & \nabla_y & \nabla_z \\ 0 & 0 & H(z) \end{vmatrix} = \hat{e}_x \partial_y H(z).$

$\underset{\substack{= H_0 \cos \omega t}}{H(z=0^+) =} \left(H = \dfrac{B}{\mu} \right)$

)

Then A_y at the boundary is 0, and the simplest solution is $A_y = 0$ at any $z > 0$.

Since $A_z = \mu_0 H(z) \, \hat{y}$, then $\nearrow \mu \cdot \hat{y} \cdot H(z,t)$ $\qquad \{ H(0,t) = H_0 \cos(\omega t).$

$$\nabla^2 A_z = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mu_0 H(z) \, \hat{y} \quad ; \quad \mu_0 \mu \sigma \frac{\partial (\mu(z)\hat y)}{\partial t} = \mu \sigma \frac{\partial A_z}{\partial t} \mu_0$$

$$\mu_0 \hat{y} \frac{\partial^2}{\partial z^2} H(z) = \mu_0 \hat{y} \; \mu \sigma \frac{\partial H(z)}{\partial t}$$

Let us try $\quad H(z,t) = h(z) \, e^{-i\omega t}$

$$\frac{\partial H(z,t)}{\partial t} = -i\omega \, h(z) \, e^{-i\omega t}$$

Then:

$$\boxed{ \left(\frac{\partial^2}{\partial z^2} + i \omega \mu \sigma \right) h(z) = 0 } \qquad\qquad (5.163)$$

Let us try $\quad h(z) = e^{ikz}$

$$\frac{\partial^2 h(z)}{\partial z^2} = (ik)^2 e^{ikz} = -k^2 e^{ikz} \qquad \text{(k can be complex in principle)}$$

Then: $\quad k^2 = i \mu \sigma \omega$

$$k = \pm \sqrt{i} \, \sqrt{\mu \sigma \omega} = \pm \sqrt{e^{i\pi/2}} \, \sqrt{\mu \sigma \omega}$$

$$= \pm \, e^{i\pi/4} \sqrt{\mu \sigma \omega} = \pm \left(\underbrace{\cos \pi/4}_{1/\sqrt{2}} + i \underbrace{\sin \pi/4}_{1/\sqrt{2}} \right) \sqrt{\mu \sigma \omega}$$

$$= \pm \frac{(1+i)}{\sqrt{2}} \sqrt{\mu \sigma \omega} \qquad\qquad (5.164)$$

$$h(z) = e^{ikz} = e^{\pm i (1+i) \sqrt{\frac{\mu \sigma \omega}{2}} \, z}$$

$$\delta \stackrel{def}{=} \sqrt{\frac{2}{\mu \sigma \omega}}$$

$$h(z) = e^{\pm(i-1)\frac{z}{\delta}}$$

the solution with (+) is physical since $h(z)$ can't grow
(-) is unphysical) with z.

$$h(z) = e^{-z/\delta} e^{i\frac{z}{\delta}}$$

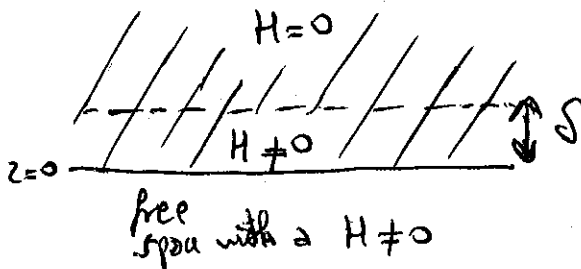$$\boxed{H(z,t) = A\, e^{-z/\delta}\, e^{i(z/\delta - \omega t)}}$$

in general we can have
a constant in front. Simply
asking that $H(z=0^+, t) = H_0\, e^{-i\omega t}$ we get

$$A = H_0$$

and then taking the real part:

$$\boxed{H(z,t) = H_0\, e^{-z/\delta} \cos(z/\delta - \omega t)} \qquad (5.166)$$

The magnetic field fall-offs exponentially
in z. Thus, it is confined to a depth "δ".

If $\sigma \to \infty$ (perfect conductor)

then $\delta \to 0$ (no fields inside a perfect conductor)

Since H varies with time, then from $\nabla \times \vec{H} = \vec{J}$

$\vec{J} = \sigma \vec{E}$

we can get the electric field:

$$\nabla \times \vec{H} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \nabla_x & \nabla_y & \nabla_z \\ H_x & 0 & 0 \end{vmatrix} = \hat{e}_y \underbrace{\nabla_z H_x}_{} - \hat{e}_z \underbrace{\nabla_y H_x}_{=0}$$

$$\underline{\frac{d}{dz}\left[H_0 e^{-z/\delta} \cos(z/\delta - \omega t)\right]}$$

$$E_y = \frac{1}{\sigma} H_0 \underbrace{\frac{d}{dz}\left[e^{-z/\delta} \cos(z/\delta - \omega t)\right]}_{}$$

$$\underbrace{-\frac{1}{\delta} e^{-z/\delta} \cos(z/\delta - \omega t)}_{Re(e^{iz/\delta} e^{-i\omega t})} - \underbrace{\frac{e^{-z/\delta}}{\delta} \sin(z/\delta - \omega t)}_{Re(-i \, e^{iz/\delta} e^{-i\omega t})}]$$

$$= +\underbrace{\frac{1}{\delta\sigma}}_{\frac{\mu\delta\omega}{2}} H_0 e^{-z/\delta} \underbrace{Re\left[-e^{iz/\delta} e^{-i\omega t} + i \, e^{iz/\delta} e^{-i\omega t}\right]}_{}$$

$$Re\left[e^{iz/\delta} e^{-i\omega t}\underbrace{(-1+i)}_{}\right]$$

$$\searrow e^{i3\pi/4} \cdot \sqrt{2} \qquad \left(\begin{array}{l} \frac{-1+i}{\sqrt{2}} \cdot \sqrt{2} = \\ = \sqrt{2} \, e^{i3\pi/4} \end{array} \right)$$

$$= \boxed{\frac{\mu\delta\omega}{\sqrt{2}} H_0 e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t + 3\pi/4\right) = E_y} \qquad (5.167)$$

$\llcorner$ not in phase with H_x.

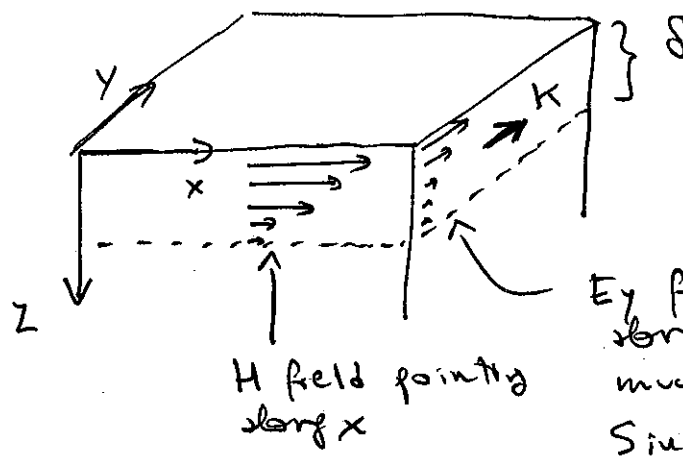If $\omega \to 0$, $\delta\omega = \sqrt{\frac{2\omega}{\mu\sigma}} \to 0$; then, in the static limit the electric field inside the conductor is 0, as expected, since a static mag-field cannot generate an electric field. The magnetic field penetrates all the way since $\delta \to \infty$ as $\omega \to 0$.

If $\omega \to \infty$, the electric fields induce $\to \infty$ as well and the mag-field cannot penetrate due to eddy currents.

The ratio $\dfrac{E_y}{cB_x} \stackrel{\uparrow}{=} \dfrac{E_y}{c\mu H_x} \propto \dfrac{\mu\delta\omega H_o}{c\mu_o H_o} = \dfrac{\mu\delta}{c}$

dimensionless

$\vec{B}=\mu\vec{H}$

in practice
this # is $\ll 1$
so the fields are
mainly magnetic.

H field pointing
along x

E_y field pointing
along y and
much smaller than B.
Since $J_y = \sigma E_y$, then along y
there is a small current, with
integral K_y

$$\boxed{K_y(t) = \int_0^\infty J_y(z,t)\,dz} = \int_0^\infty \sigma\,\frac{\mu\omega\delta}{\sqrt{2}}\,H_o\,\bar{e}^{-z/\delta}\cos\left(\frac{z}{\delta} - \omega t + \frac{3\pi}{4}\right)dz$$
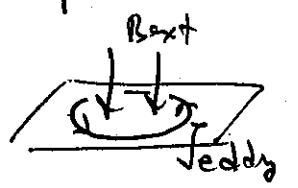
$$\stackrel{\uparrow}{=} \underbrace{\frac{\sigma\mu\omega\delta}{\sqrt{2}}\,H_o}\,\delta\int_0^\infty du\,\bar{e}^{u}\cos(u - \omega t + 3\pi/4) =$$

$\frac{z}{\delta} = u$

$\dfrac{\sigma\mu\omega}{\sqrt{2}}\,H_o\,\dfrac{2}{\kappa\delta\omega}$

$= H_o\sqrt{2}\cdot\left(-\frac{1}{\sqrt{2}}\cos\omega t\right)$

(next page) $= -H_o\cos\omega t$

From $\nabla\times\vec{H}=\vec{J}$
$[H] = [\frac{J}{L}]$
Thus
$[K] = [J\cdot L]$
$= [A][\frac{1}{L}][L]$
$=$

These are the "eddy currents" that are generated
in the metal. In chapter 8, the discussion continues.
Basically these currents tend to generate a magnetic
field that opposes the external magnetic field.
These currents also generate heat (called "induction
heat")

$$\int_0^\infty dx\ \bar{e}^{-x}\cos(x-\alpha) = \int_0^\infty dx\ e^{-x}\left(\frac{e^{i(x-\alpha)} + e^{-i(x-\alpha)}}{2}\right) =$$

$$= \frac{1}{2}e^{-i\alpha}\underbrace{\int_0^\infty dx\ e^{(-1+i)x}}_{\dfrac{e^{(-1+i)x}}{(-1+i)}\Big|_0^\infty} + \frac{e^{i\alpha}}{2}\underbrace{\int_0^\infty dx\ e^{(-1-i)x}}_{\dfrac{e^{(-1-i)x}}{(-1-i)}\Big|_0^\infty} =$$

$$= \frac{1}{2}e^{-i\alpha}\frac{(-1)}{(-1+i)} + \frac{e^{i\alpha}}{2}\frac{(-1)}{(-1-i)} = \uparrow$$

$$-1+i = \sqrt{2}\ e^{i\,3\pi/4}$$

$$-1-i = \sqrt{2}\ e^{-i\,3\pi/4}$$

$$= \frac{1}{2}(-1)\frac{e^{-i(\omega t - 3\pi/4)}}{\sqrt{2}\ e^{i3\pi/4}} + \frac{(-1)}{2}\frac{e^{i(\omega t - 3\pi/4)}}{\sqrt{2}\ e^{-i3\pi/4}}$$

$$= -\frac{1}{2\sqrt{2}}e^{-i\omega t} - \frac{1}{2\sqrt{2}}e^{i\omega t} = -\frac{1}{\sqrt{2}}\cos(\omega t)$$
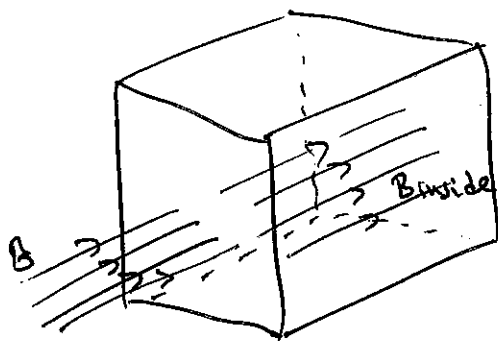
Summary

Consider a real metal with σ finite.

If $\omega = 0$ (static magnetic fields), then $\delta = \sqrt{\dfrac{2}{\mu \sigma \omega}} \rightarrow \infty$.
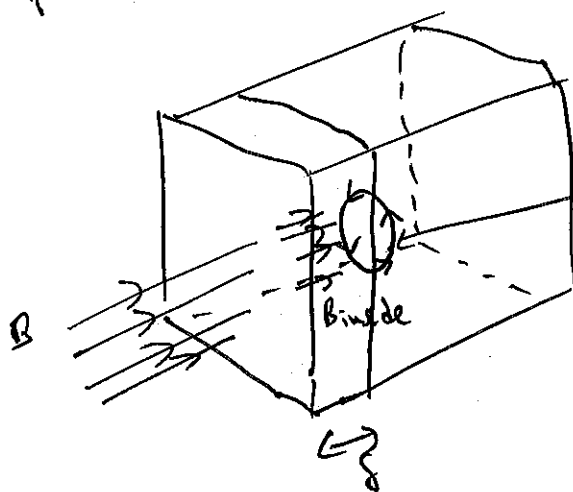
the magnetic field can penetrate in the entire sample.

$$E \propto \delta \cdot \omega = \sqrt{\frac{2\omega}{\mu \sigma}} \xrightarrow[\omega \to 0]{} 0$$
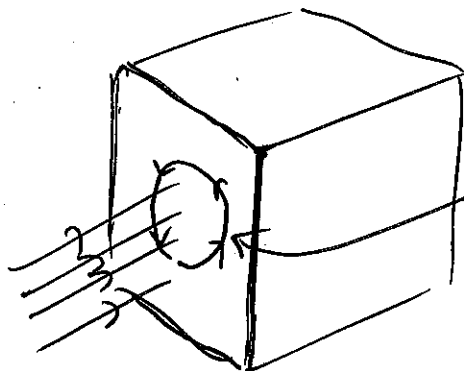
No electric field
generated by static B
as expected.

B → B inside

If $\omega = $ finite, then δ is finite as well, and E is finite, thus J is finite:

B → B inside

Eddy generated
are generated and
they prevent B
from penetrating beyond "δ".

If $\omega \to \infty$, then $\delta \to 0$, and no magnetic field can penetrate. this can be achieved by $|\vec{E}| \to \infty$, i.e. $|\vec{J}| \to \infty$ at the surface.
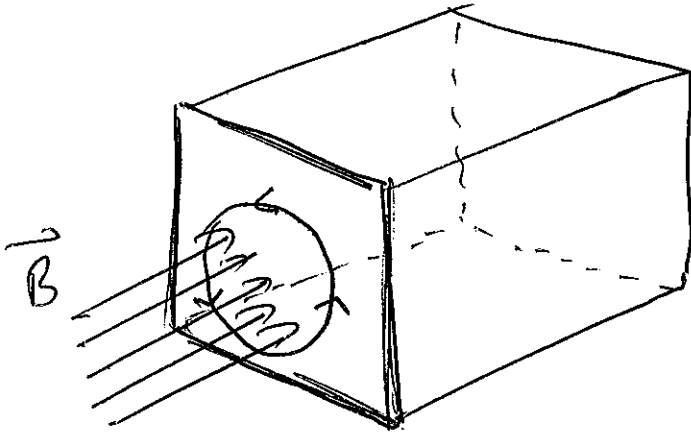

surface current
prevents B from
getting in.

What happens if conductor is perfect $\sigma \to \infty$?

) For $\omega = $ finite, $\delta \to 0$ and the magnetic field does **not** penetrate. The electric field

$$E \sim \sqrt{\frac{2\omega}{\mu\sigma}} \to 0 \quad \text{inside.}$$

For $\sigma = \infty$, $\omega = $ finite the magnetic field cannot penetrate the material.
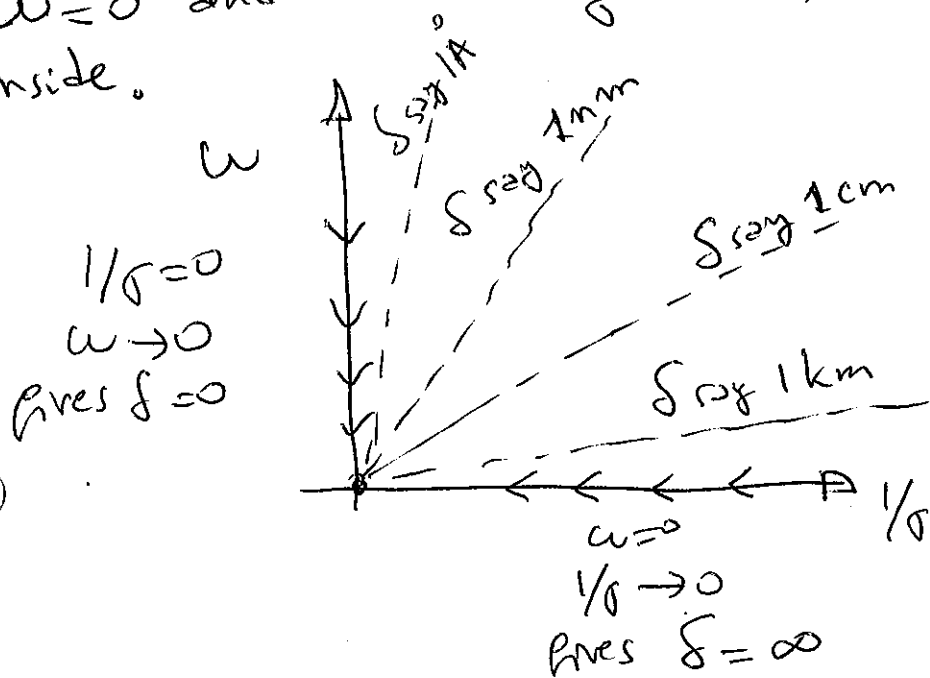

$\sigma = \infty$

$\omega > 0$

The surface current induce by Faraday's law kills B inside.

)

The case studied on the previous page is more realistic: all conductors have a finite σ while ω can be truly turned from nonzero to zero. Then, a __static__ magnetic field is expected to penetrate into a metal. There will be a magnetization $\vec{M}$ but it will be small:

$$\vec{M} \underset{\text{def}}{=} \frac{\vec{B}}{\mu_0} - \vec{H} = \frac{\vec{B}}{\mu_0} - \frac{\vec{B}}{\mu} = \frac{(\mu - \mu_0)\vec{B}}{\mu\mu_0}$$

and $\mu - \mu_0$ is almost 0 for a metal.

)

The σ finite case is also realistic in the following gedanken experiment. We introduce a magnetic field (static) inside a metal at high T where σ is not large. Then, we cool down and suppose we cross a phase transition to a state with a high σ. In this case the magnetic field remains inside the metal, i.e. it is not expelled. If the transition is to a superconductor, then the Meissner effect kicks in and $\vec{B}$ is expelled but that effect is not contained in the Max. Eqs.

In the plane $\omega - 1/\sigma$ the limit $\omega = 0$, $1/\sigma = 0$ is singular. If we fix $\frac{1}{\sigma} = 0$ and then $\omega \to 0$, the field does not penetrate the metal. If we fix $\omega = 0$ and then do $\frac{1}{\sigma} \to 0$, the field remains inside.

Since $\delta \sim \frac{1}{\sqrt{\omega \sigma}}$, the lines of "constant δ" are given by $\frac{1}{\sqrt{\omega \sigma}} = A$ or $\omega = \frac{1}{A^2 \sigma}$. Examples are shown as dashed lines.

ω

δ say $\frac{1}{A}$

δ say 1 mm

δ say 1 cm

δ say 1 km

$1/\sigma = 0$
$\omega \to 0$
gives $\delta = 0$

$1/\sigma$

$\omega = 0$
$1/\sigma \to 0$
gives $\delta = \infty$