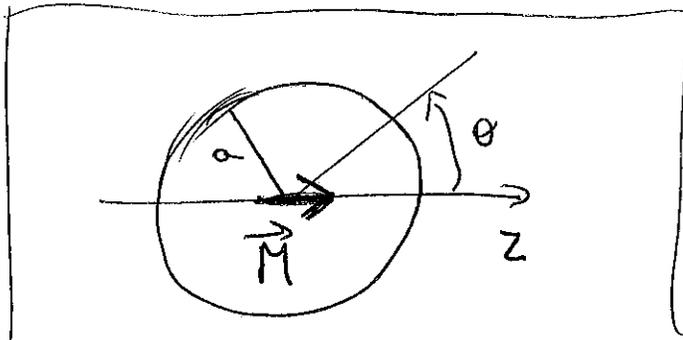


5.10 Example: uniformly magnetized sphere



\vec{M} assumed uniform inside sphere and permanent

Let us use method C(2) (page 196)

$$\vec{M} = M_0 \vec{e}_3 \quad \text{magnitude of magnetization}$$

$$\sigma_M = \vec{n} \cdot \vec{M} = M_0 \cos \theta \quad \text{since } \vec{n} = \vec{e}_3 \cos \theta + \vec{e}_\perp \sin \theta$$

Then, from Eq (5.100) we use only the second term since the first term is 0 since $\nabla' \cdot \vec{M} = 0$ inside the sphere.

$$\Phi_M = \frac{1}{4\pi} \oint_{\text{sphere}} \frac{\vec{n}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} da'$$

$$= \frac{1}{4\pi} \int \frac{M_0 \cos \theta'}{|\vec{x} - \vec{x}'|} \omega^2 d\Omega'$$

Due to the symmetry of the problem, let us use (3.70)

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{4\pi}{4\pi} \sum_{l,m} \frac{1}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

Since $\int d\Omega'$ contains $\int_0^{2\pi} d\phi'$ then $m \neq 0$ is 0, since

$$\int_0^{2\pi} e^{im\phi'} d\phi' = 0, \text{ if } m \neq 0, \text{ etc.}$$

~~Thus~~ Or, equivalently, $\cos \theta' = \sqrt{\frac{4\pi}{3}} Y_{10}$ from page 103

Then:

$$\Phi = \frac{4\pi}{4\pi} \sum_{l,m} \frac{M_0 a^2}{4\pi} \frac{1}{r_{>}^{l+1}} \int d\Omega' \underbrace{Y_{10}(\theta', \phi') \sqrt{\frac{4\pi}{3}}}_{\cos \theta'} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

Eq. (3.55) says

$$\int d\Omega' Y_{l'm'}^*(\theta', \phi') Y_{lm}(\theta', \phi') = \delta_{l'l} \delta_{m'm}$$

$$\text{Then: } \int_{l,m} M_0 a^2 \frac{r^l}{r^{l+1}} \frac{1}{(2l+1)} Y_{lm}(\theta, \phi) \sqrt{\frac{4\pi}{3}} d\Omega \int_{m,0} =$$

$$= M_0 a^2 \frac{r^l}{r^2} \frac{1}{3} \underbrace{Y_{10}(\theta, \phi)}_{\sqrt{\frac{3}{4\pi}} \cos\theta} \sqrt{\frac{4\pi}{3}}$$

$$= \boxed{\frac{M_0 a^2}{3} \frac{r}{r^2} \cos\theta} \quad (\text{S.104})$$

Inside sphere, $r < r$, $r > a$

$$\left[\begin{array}{l} \text{inside} \\ \Phi_M = \frac{M_0 r}{3} \cos\theta = \frac{M_0 z}{3} \\ \Phi_M^{\text{outside}} = \frac{M_0 a^3}{3 r^2} \cos\theta = \frac{M_0}{3} \left(\frac{a}{r}\right)^3 z \end{array} \right]$$

\uparrow
 $r < a$
 $r > r$

Inside the sphere

$$\Phi_M = \frac{1}{3} \mu_0 z \quad (5.81)$$

$$H_z = -\frac{\partial \Phi_M}{\partial z} = -\frac{\mu_0}{3}$$

↑
antiparallel to \vec{M}

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

$$\vec{B} = \mu_0 (\vec{H} + \vec{M})$$

$$\uparrow = \mu_0 \left(\frac{\vec{M}}{3} + \vec{M} \right) = \mu_0 \frac{2}{3} \vec{M}$$

↑
parallel to \vec{M}

Outside sphere

$$\Phi_M = \frac{1}{3} \mu_0 a^3 \frac{z}{r^3}$$

which is the potential of a dipole moment
of value

$$\vec{m} = \frac{4\pi a^3}{3} \vec{M}$$

($\frac{4\pi a^3}{3}$ = volume sphere)

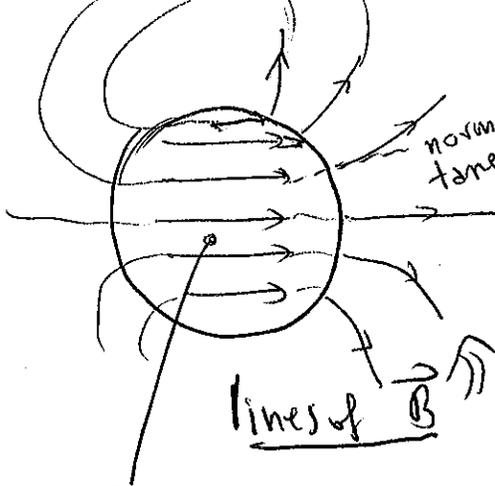
(see bottom of page 196:

$$\Phi_M \simeq \frac{\vec{m} \cdot \vec{r}}{4\pi r^3} = \frac{1}{3} \mu_0 a^3 \frac{z}{r^3}$$

$$\text{then } \vec{m} = m_z \vec{e}_z$$

$$m_z = \frac{4\pi}{3} \mu_0 a^3 M$$

There are no higher multipoles,
which is interesting.



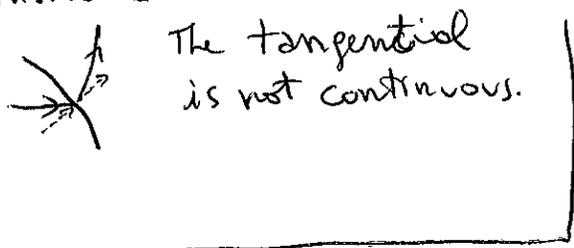
Since $\nabla \cdot \vec{B} = 0$, the lines are closed (no source)

Outside

$$\begin{aligned} \vec{B}_{out} &= \mu_0 (\vec{H}_{out} + \vec{M}_{out}) = 0 \\ &= \mu_0 \vec{H}_{out} \\ &= -\mu_0 \nabla \Phi_M^{out} \\ &= -\mu_0 \nabla \left(\frac{1}{3} M_0 a^3 \frac{\cos \theta}{r^2} \right) \\ &= -\frac{\mu_0 M_0 a^3}{3} \nabla \left(\frac{\cos \theta}{r^2} \right) \end{aligned}$$

\vec{B} inside is uniform
 $\frac{2\mu_0 \vec{M}}{3}$

\vec{B} at surface has normal the same inside and outside

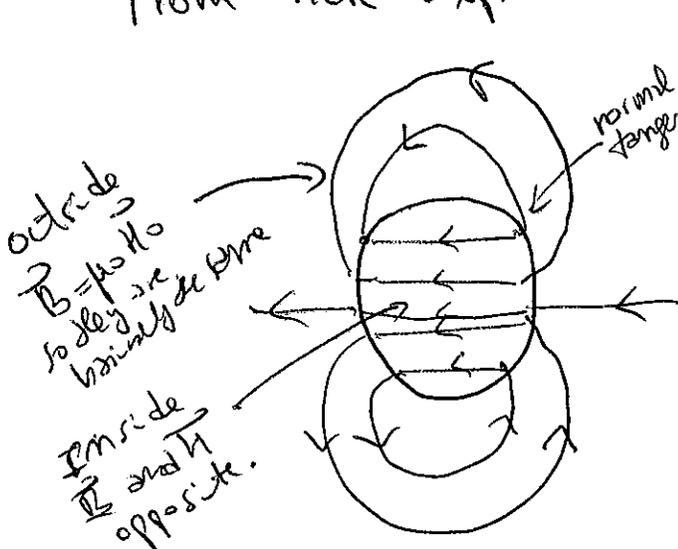


back of book

$$\begin{aligned} \nabla \psi &= \vec{e}_r \frac{\partial \psi}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial \psi}{\partial \theta} \\ &+ \vec{e}_\phi \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi} \end{aligned}$$

$$\begin{aligned} \nabla \left(\frac{\cos \theta}{r^2} \right) &= \vec{e}_r \left(\frac{-2 \cos \theta}{r^3} \right) + \vec{e}_\theta \frac{1}{r} \left(\frac{-\sin \theta}{r^2} \right) \\ &+ \vec{e}_\phi \frac{1}{r \sin \theta} \frac{\partial \Phi_M}{\partial \phi} \\ &= 0 \end{aligned}$$

From here explain draw Fig. 5.11



Since $\nabla \cdot \vec{B} = 0$, $\nabla \cdot (\vec{H} + \vec{M}) = 0$

Thus $\nabla \cdot \vec{H} = -\nabla \cdot \vec{M} \neq 0$ at surface.

Then \vec{H} has "sources" at (no closed lines) surface.

5.11 Magnetic sphere in an external field B_0

We simply "superpose" the previous problem, where the fields \vec{B} and \vec{H} of a sphere with magnetization \vec{M} ~~was~~ ^{were} found, with ~~an~~ an external field:

$$\vec{B}_{\text{inside}} = \underbrace{\frac{2\mu_0}{3} \vec{M}}_{(5.105)} + \vec{B}_0$$

$$\vec{H}_{\text{inside}} = \underbrace{-\frac{\vec{M}}{3}}_{(5.105)} + \frac{\vec{B}_0}{\mu_0}$$

Now consider the case where \vec{M} is not permanent but it is induced by \vec{B}_0 . Then, let us use

$$\vec{B}_{\text{inside}} = \mu \vec{H}_{\text{inside}}$$

$$\vec{B}_0 + \frac{2\mu_0}{3} \vec{M} = \mu \left(-\frac{\vec{M}}{3} + \frac{\vec{B}_0}{\mu_0} \right)$$

$$\vec{M} = \frac{\left(\frac{\mu}{\mu_0} - 1 \right) \vec{B}_0}{\frac{2}{3} \mu_0 + \frac{\mu}{3}}$$

$$= \boxed{\frac{3}{\mu_0} \left(\frac{\mu - \mu_0}{\mu + 2\mu_0} \right) \vec{B}_0}$$

↑ analogous of \vec{P} of a sphere in electric field \vec{E}_0
(4.57)

If the sphere is a permanent ferromagnet then

) the analysis is more complicated since we cannot use $\vec{B}_{in} = \mu \vec{H}_{in}$. In fact, eliminating \vec{M} we

get:

$$(\vec{B}_{inside} - \vec{B}_0) \frac{3}{2\mu_0} = \left(\vec{H}_{inside} - \frac{\vec{B}_0}{\mu_0} \right) (-3)$$

or

$$\frac{3}{2\mu_0} \vec{B}_{inside} + 3 \vec{H}_{inside} = \vec{B}_0 \frac{3}{\mu_0} + \vec{B}_0 \frac{3}{2\mu_0} = \frac{9}{2} \frac{\vec{B}_0}{\mu_0}$$

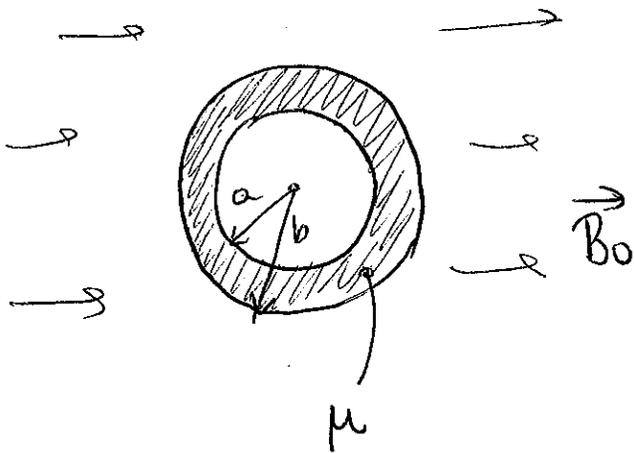
or

$$\boxed{\vec{B}_{inside} + 2\mu_0 \vec{H}_{inside} = 3\vec{B}_0} \quad (5.116)$$

The other relation needed to find \vec{B}_{inside} and \vec{H}_{inside} must come from experiments (hysteresis loop).

The discussion is left to the students to read from book.

S.12 Another example: magnetic shielding.



A spherical shell is located in a uniform magnetic field \vec{B}_0 .

Find \vec{B} and \vec{H} everywhere.

Since $\vec{J}=0$, then $\vec{H} = -\nabla\Phi_M$ (S.93)

Since $\vec{B} = \mu\vec{H}$, then $\nabla \cdot \vec{B} = 0$ means $\nabla \cdot \vec{H} = 0$ in each region.

Thus, inside each of the 3 regions we have

$$\nabla \cdot \vec{H} = \boxed{-\nabla^2 \Phi_M = 0} \text{ i.e. } \Phi_M \text{ satisfies Laplace's eq.}$$

We can then use Sec. 3.1, page 95.

Moreover, this problem has azimuthal symmetry

thus $\underline{m=0}$ (see page 101) and

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos\theta) \text{ in general}$$

Consider first $r > b$: then I must drop " r^l " terms that diverge.

$$\Phi_M = \underbrace{-H_0 r \cos \theta}_{\text{simple}} + \sum_{l=0}^{\infty} \frac{\alpha_l}{r^{l+1}} P_l(\cos \theta) \quad \begin{array}{l} \text{changing} \\ \text{notation} \\ Bl \rightarrow \alpha_l \end{array}$$

$-\frac{\partial}{\partial z}(H_0 r \cos \theta) = H_0$
 i.e. this term is sufficient to take care of the external field as $r \rightarrow \infty$
 i.e. Φ_M as given converges to the right answer as $r \rightarrow \infty$

In the inner region $a < r < b$:

$$\Phi_M = \sum_{l=0}^{\infty} \left(\beta_l r^l + \frac{\gamma_l}{r^{l+1}} \right) P_l(\cos \theta)$$

At $r < a$, I must drop the $\frac{1}{r^{l+1}}$ term that $\rightarrow \infty$ as $r \rightarrow 0$

$$\Phi_M = \sum_{l=0}^{\infty} \delta_l r^l P_l(\cos \theta)$$

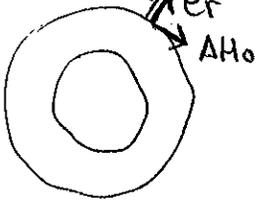
Now we have to use the boundary conditions.

From $(\vec{B}_2 - \vec{B}_1) \cdot \vec{n} = 0$ (S. 86)

we deduce B_r must be continuous since $\vec{n} = \vec{e}_r$
 that

From $\vec{n} \times (\vec{H}_2 - \vec{H}_1) = 0$ (S. 87),

we deduce $\vec{e}_r \times (\vec{H}_2 - \vec{H}_1) = 0$



$\Delta \vec{H}$ that only has components in \hat{r} and $\hat{\theta}$ since there is no $\hat{\phi}$ dependence in Φ_M .

Then $\vec{e}_r \times \Delta \vec{H} = 0$ means

$$\begin{vmatrix} e_r & e_\theta & e_\phi \\ 1 & 0 & 0 \\ \Delta H_r & \Delta H_\theta & 0 \end{vmatrix} = \vec{e}_\phi \Delta H_\theta = 0 \quad \text{i.e.} \quad \Delta H_\theta = 0$$

So H_θ is continuous

$$\left. \frac{\partial \Phi_M}{\partial \theta} \right|_b^{r>b} = \left. \frac{\partial \Phi_M}{\partial \theta} \right|_b^{a<r<b} ; \quad \left. \frac{\partial \Phi_M}{\partial \theta} \right|_a^{r<a} = \left. \frac{\partial \Phi_M}{\partial \theta} \right|_a^{a<r<b}$$

$$\begin{aligned} & \left. \frac{\partial \Phi_M}{\partial \theta} \right|_b^{r>b} = -H_0 b (-\sin \theta) + \sum_{l=0}^{\infty} \frac{\alpha_l}{b^{l+1}} \frac{\partial P_l(\cos \theta)}{\partial \theta} \Big|_{\theta=0} \\ & \quad \quad \quad \frac{\partial(\cos \theta)}{\partial \theta} = \frac{\partial P_l(\cos \theta)}{\partial \theta} \Big|_{\theta=0} \\ & = \sum_{l=0}^{\infty} \left(\beta_l b^l + \frac{\alpha_l}{b^{l+1}} \right) \frac{\partial P_l(\cos \theta)}{\partial \theta} \\ & \left. \frac{\partial \Phi_M}{\partial \theta} \right|_a^{r<a} = \sum_{l=0}^{\infty} \delta_l a^l \frac{\partial P_l(\cos \theta)}{\partial \theta} = \sum_{l=0}^{\infty} \left(\beta_l a^l + \frac{\alpha_l}{a^{l+1}} \right) \frac{\partial P_l(\cos \theta)}{\partial \theta} \end{aligned}$$

Then:

$$-H_0 b + \frac{\alpha_1}{b^2} = \beta_1 b + \frac{\gamma_1}{b^2}$$

$$\frac{\alpha_l}{b^{l+1}} = \beta_l b^l + \frac{\gamma_l}{b^{l+1}} \quad (l \neq 1)$$

$$\delta_1 a^{\frac{1}{2}} = \beta_1 a + \frac{\gamma_1}{a^2}$$

$$\delta_l a^l = \beta_l a^l + \frac{\gamma_l}{a^{l+1}} \quad (l \neq 1)$$

The other equations are (Br continuous):

$$\mu_0 \left. \frac{\partial \Phi_M}{\partial r} \right|_b^{r>b} = \mu_0 \left. \frac{\partial \Phi_M}{\partial r} \right|_b^{a<r<b}$$

$$\mu_0 \left. \frac{\partial \Phi_M}{\partial r} \right|_a^{r<a} = \mu_0 \left. \frac{\partial \Phi_M}{\partial r} \right|_a^{a<r<b}$$

$$\mu_0 \left[-H_0 \cos \theta + \sum_{l=0}^{\infty} \frac{[-(l+1)\alpha_l}{b^{l+2}} P_l(\cos \theta) \right] =$$

$$= \mu_0 \left[\sum_{l=0}^{\infty} \left(l \beta_l b^{l-1} + \gamma_l \frac{(-l-1)}{b^{l+2}} \right) P_l(\cos \theta) \right]$$

$$\mu_0 \sum_{l=0}^{\infty} \delta_l a^{l-1} P_l(\cos \theta) =$$

$$= \mu_0 \sum_{l=0}^{\infty} \left(l \beta_l a^{l-1} + \gamma_l \frac{(-l-1)}{a^{l+2}} \right) \cdot P_l(\cos \theta)$$

Then:

$$\mu_0 \left(-H_0 - \frac{2}{b^3} \alpha_1 \right) = \mu_0 \left(\beta_1 - \frac{2\gamma_1}{b^3} \right) \quad l=1$$

$$\mu_0 \frac{(-l-1) \alpha_l}{b^{l+2}} = \mu \left(l \beta_l b^{l-1} - \frac{(l+1)\gamma_l}{b^{l+2}} \right) \quad l \neq 1$$

$$\mu_0 \delta_1 = \mu_0 \left(\beta_1 - \frac{2\gamma_1}{a^3} \right) \quad l=1$$

$$\mu_0 \delta_l a^{l-1} = \mu \left(l \beta_l a^{l-1} - \frac{(l+1)\gamma_l}{a^{l+2}} \right) \quad l \neq 1$$

For $l=1$ we have

$$-H_0 b^3 + \alpha_1 = \beta_1 b^3 + \gamma_1$$

$$\delta_1 a^3 = \beta_1 a^3 + \gamma_1$$

$$\mu_0 (-H_0 b^3 - 2\alpha_1) = \mu (\beta_1 b^3 - 2\gamma_1)$$

$$\mu_0 \delta_1 a^3 = \mu (\beta_1 a^3 - 2\gamma_1)$$

$$\alpha_1 - b^3 \beta_1 - \gamma_1 = b^3 H_0$$

$$2\alpha_1 + \mu' b^3 \beta_1 - 2\mu' \gamma_1 = -b^3 H_0$$

$$a^3 \beta_1 + \gamma_1 - \delta_1 a^3 = 0$$

$$\mu' \beta_1 a^3 - 2\mu' \gamma_1 - a^3 \delta_1 = 0$$

(S.120)

$$\left(\mu' = \frac{\mu}{\mu_0} \right)$$

I will not show it, but for $l \neq 1$ all the coefficients vanish. Then, $l=1$ is all we have to handle.

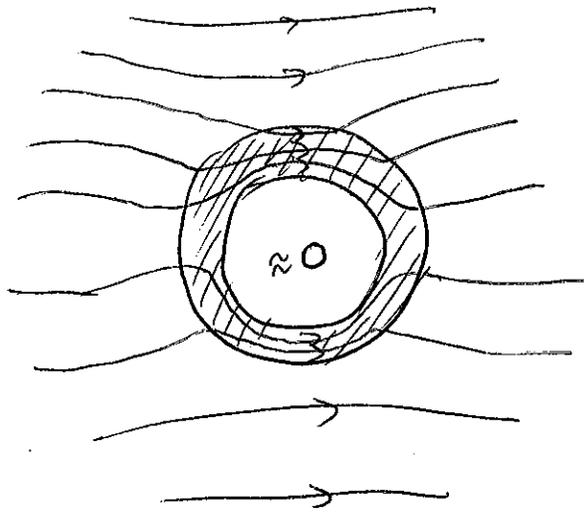
I will not show it either, but in the limit $\mu^1 \gg 1$, i.e. $\mu \gg \mu_0$, it can be shown that

$$\alpha_1 \rightarrow b^3 H_0$$

$$\beta_1 \propto \frac{1}{(\mu/\mu_0)} \xrightarrow{\mu/\mu_0 \rightarrow \infty} 0$$

From (S.121),
 $\beta_1 \sim \frac{\mu^1}{\mu^2} H_0 \sim \frac{1}{\mu^1} H_0$
 $\sim \frac{1}{(\mu/\mu_0)} H_0$

Then inside the spherical shell there is no magnetic field and it can be used for shielding.



The "lines of B" like to take advantage of a high μ in the shell.

$\frac{\mu}{\mu_0}$ can be as large as 1,000, or more.

"magnetic shielding"