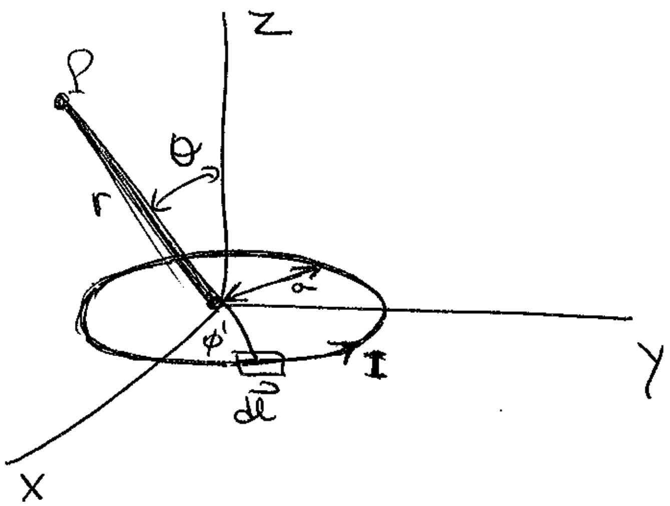
5.5 Example



$$J_\phi = \frac{I}{a} \cdot \delta(\cos\theta')\,\delta(r'-a)$$

needed by dim. analysis

Note: in $\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$ if note that dx has length L units. Thus, $\delta(x)$ has $\frac{1}{L}$ units.

Thus: $[\delta(r'-a)] = \frac{1}{L}$

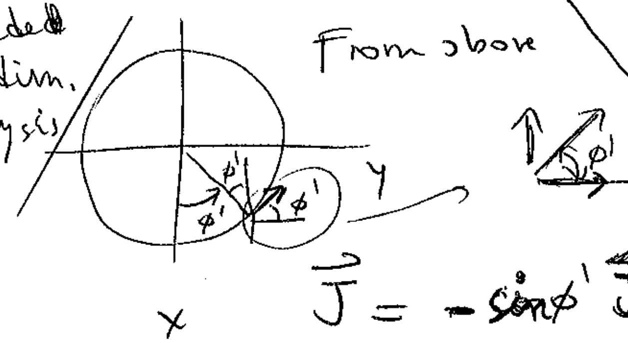
but for the angle $d\theta'$ is dimensionless and $[\delta(\cos\theta')] \equiv \frac{1}{L^0}$.

$J_r = 0$

$J_\theta = 0$

only J_ϕ is $\neq 0$.

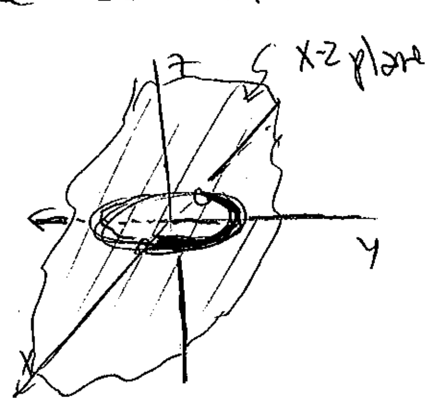
From above



$$\vec{J} = -\sin\phi'\, J_\phi\, \vec{i} + \cos\phi'\, J_\phi\, \vec{j}$$

This is a cylindrically ~~geometry~~ symmetric geometry, thus we can choose a particular plane to do the calculation. We choose $\phi = 0$, i.e. the x-z plane.

So "$\vec{x}$" will be in the x-z plane.



The "x" component of $\vec{J}(\vec{x}')$ has a $\underline{\underline{\sin\phi'}}$ which leads to a cancellation in $\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$ (5.32) because $\int_0^{2\pi} d\phi' \frac{\sin\phi'}{f(\cos\phi)}$

$$|\vec{x}-\vec{x}'| \to \sqrt{r^2+r'^2-2rr'(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi')}$$

$\cos\pi/2=0$, $\sin\pi/2=1$

will cancell (since say ϕ' and $-\phi'$ will cancell in the integral)
or $\sin\phi'$ is odd under $\phi'\to-\phi'$
$\cos\phi'$ is even under $\phi'\to-\phi'$

Then, we only keep the "y" component in $\vec{J}'(\vec{x}')$:

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\frac{I}{a}\int \underbrace{r'^2dr'd\Omega'}_{d^3x'} \cdot \cos\phi' \cdot \frac{\delta(\cos\theta')\,\delta(r'-a)}{|\vec{x}-\vec{x}'|} =$$

in x-z plane A_ϕ points along y.

From $\vec{J}$ decomposed in $\vec{i}$ and $\vec{j}$

from J_ϕ

$$= \boxed{\frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{\cos\phi'd\phi'}{\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}}} = \quad (5.36)$$

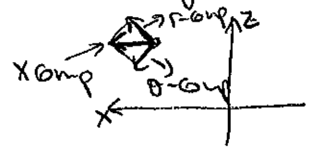
$\sin\theta'=1$
$\cos\theta'=0 \to \delta$ function $\theta'=\pi/2$

= this integral can be expressed in terms of "complete elliptical integrals K and E"
(we will not do this explicitly)

To get the field $\vec{B}$, we have to get $\nabla \times \vec{A}$.

But we said the "x" component was 0 and in the plane x-z, the "x" component is a linear combination of A_θ and A_r, then $A_r = 0, A_\theta = 0$.

~~Also since $\vec{J}$ has components only along "$\vec{x}$" and "$\vec{y}$", then $A_\theta = 0$ as well.~~



We go to the back of the book and in $\nabla \times \vec{A}$ with spherical coordinates, we look for "$A_3 \neq 0$"; $A_1 = A_2 = 0$.

$$\nabla \times \vec{A} = \vec{e}_1 \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta\, A_3)$$

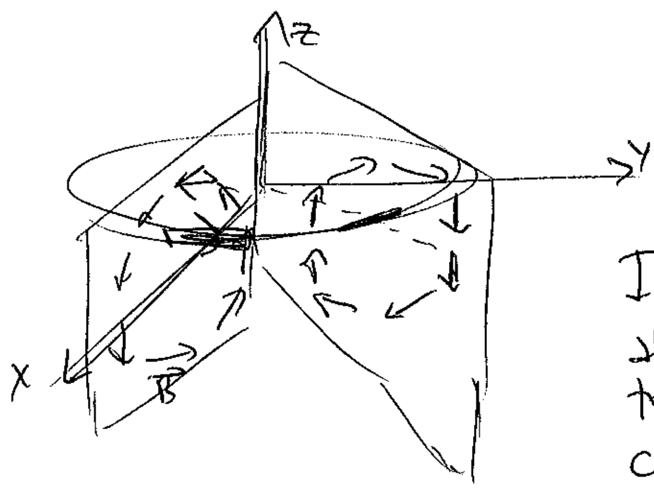
$$\vec{e}_2 \left(-\frac{1}{r} \frac{\partial}{\partial r} (r A_3) \right) + \vec{e}_3 \cdot 0$$

Then, for this particular problem:

$$\boxed{\begin{aligned} B_r &= \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta\, A_\phi) \\ B_\theta &= -\frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \\ B_\phi &= 0 \end{aligned}}$$

Intuitively we can understand the ϕ cancellation:

) For each small segment of current I, the thumb rule says the $\vec{B}$ has to be as shown:



I.e. the vector $\vec{B}$ are always on a plane normal to the small segment that carries I. Thus, B_ϕ is 0.

) It is also interesting to study the limit $r \gg a$. In (5.36), if $\sqrt{a^2+r^2-2ra\sin\theta\cos\phi} \approx \sqrt{r^2} = r$ ~~results~~ then $\int_0^{2\pi} \cos\phi'\, d\phi'$ gives 0 ~~y por lo tanto se requiere el término siguiente:~~ and for this reason we need the next term:

$$\frac{1}{\sqrt{a^2+r^2-2ra\sin\theta\cos\phi'}} = \frac{1}{r\sqrt{1+\underbrace{\frac{a^2}{r^2}-\frac{2a}{r}\sin\theta\cos\phi'}_{u}}} \approx \frac{1}{r}\left(1-\frac{1}{2}u\right)$$

$$= \frac{1}{r}\left(1-\frac{1}{2}\left(-\frac{2a}{r}\sin\theta\cos\phi'\right)\right)$$

$$) = \frac{1}{r}\left(1+\frac{a}{r}\sin\theta\cos\phi'\right)$$

$$A_\phi \cong \frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{\cos\phi'\, d\phi'}{r}\left(1+\frac{a}{r}\sin\theta\cos\phi'\right) =$$

this term gives 0 as already discussed

~~este término da 0 como ya se discutió~~

$$= \frac{\mu_0 I a^2}{4\pi r^2}\sin\theta \underbrace{\int_0^{2\pi}\cos^2\phi'\, d\phi'}_{\int_0^{2\pi}\left(\frac{1+\cos 2\phi}{2}\right)d\phi' = \pi} = \frac{\mu_0 I a^2}{4\pi r^2}\sin\theta\, \frac{1}{2}\, 2\pi = \left(\frac{\mu_0}{4\pi}\right)\frac{(I\pi a^2)\sin\theta}{r^2}$$

$$\int_0^{2\pi}\cos 2\phi'\, d\phi' = 0$$

$$\cos 2\phi = \cos^2\phi - \sin^2\phi = 2\cos^2\phi - 1$$

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\ \frac{\mu_0 I a^2}{4\pi r^2}\sin\theta\ \pi\right]$$

$$= \frac{\mu_0 I a^2}{4 r^3}\,\frac{1}{\sin\theta}\underbrace{\frac{\partial}{\partial\theta}(\sin^2\theta)}_{2\sin\theta\cos\theta} = \boxed{\frac{\mu_0(I a^2\pi)\cos\theta}{2\pi r^3}} \quad (5.41)$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi) = -\frac{1}{r}\,\frac{\mu_0 I a^2}{4\pi}\sin\theta\ \pi\ \underbrace{\frac{\partial}{\partial r}\left(r\frac{1}{r^2}\right)}_{} =$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2}$$

$$= \boxed{\frac{\mu_0(I\pi a^2)}{4\pi r^3}\sin\theta} \quad (5.41)$$

The $\frac{1}{r^3}$ dependence shows that the magnetic fields at $r \gg a$ are "dipolar in character"

with a magnetic dipole moment $\boxed{m = \pi I a^2}$.

Note that the $\frac{1}{r^2}$ contribution cancelled because there are no magnetic monopoles.

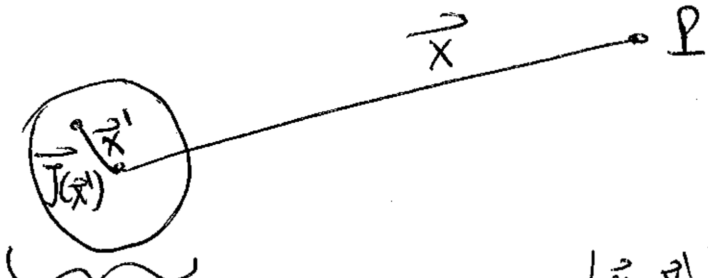
See (4.12) for the case of electric fields

$$E_r = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

$$= \frac{1}{2\pi\epsilon_0} p \frac{\cos\theta}{r^3}$$

dipole pointing along the z-axis.

5.6 Magnetic fields of a localized Current distribution. Magnetic moments



currents localized here

$$\left|\vec{n} - \frac{\vec{x}'}{|x|}\right| = \sqrt{\left(\vec{n} - \frac{\vec{x}'}{|x|}\right)\cdot\left(\vec{n} - \frac{\vec{x}'}{|x|}\right)} \approx \left(1 - 2\frac{\vec{n}\cdot\vec{x}'}{|x|}\right)^{1/2}$$

$$\vec{x} = \vec{n}\,|\vec{x}|$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|\left|\vec{n} - \frac{\vec{x}'}{|\vec{x}|}\right|} \approx \frac{1}{|\vec{x}|}\left(1 + \frac{\vec{n}\cdot\vec{x}'}{|\vec{x}|}\right) = \frac{1}{|\vec{x}|}\left(1 + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2}\right) =$$

$$= \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3}$$

$\vec{n}$ = normal vector in $\vec{x}$ direction

$\vec{x} = |\vec{x}|\,\vec{n}$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|} \approx \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\left(\frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3}\right) d^3x'$$

or by components:

$$A_i(\vec{x}) \approx \frac{\mu_0}{4\pi}\left[\frac{1}{|\vec{x}|}\int J_i(\vec{x}')\,d^3x' + \frac{\vec{x}\cdot}{|\vec{x}|^3}\int J_i(\vec{x}')\,\vec{x}'\,d^3x'\right]$$

Consider

$$\int \rho \vec{J} \cdot \nabla' f \, d^3x' \quad \text{where } \rho = \rho(\vec{x}'),\ f = f(\vec{x}')$$

Let us integrate by parts $\int d(uv) = \int u\,dv + \int v\,du$ as we did a few times before:

$$\nabla' \cdot (\underbrace{\rho \vec{J}}_{u} \underbrace{f}_{v}) = [\nabla' \cdot (\rho \vec{J})] f + \rho \vec{J} \cdot \nabla' f$$

Then

$$\int \rho \vec{J} \cdot \nabla' f \, d^3x' = \underbrace{\int \nabla' \cdot (\rho \vec{J} f) \, d^3x'}_{\xrightarrow[\text{if all localized}]{} 0} - \int [\nabla' \cdot (\rho \vec{J})] f \, d^3x' =$$

$$= -\int \rho f (\nabla' \cdot \vec{J}) \, d^3x - \int f \vec{J} \cdot \nabla' \rho \, d^3x' \qquad \text{which is (5.52)}$$

$$\nabla' \cdot (\rho \vec{J}) = \rho (\nabla' \cdot \vec{J}) + \vec{J} \cdot (\nabla' \rho)$$

If we take $\nabla' \cdot \vec{J} = 0$ as expected from localized currents and $f = 1$ and $\rho = x'_i$ then: $\nabla' f = 0$, $\nabla' \cdot \vec{J} = 0$ and only one term remains, which is equal to 0.

$$\int f \vec{J} \cdot \nabla' \rho \, d^3x = \int \vec{J} \cdot \underbrace{\nabla' x'_i}_{= \vec{e}_i \cdot 1} \, d^3x' = \int J_i \, d^3x'$$

i.e. it just gives a unit vector in the "i" direction

Then, we have arrived to the mathematical proof that there is no monopole term in magnetostatics, since the coefficient cancells:

$$\boxed{\int J_i(\vec{x}\,')\, d^3x' = 0} \qquad \left(\text{or } \int \vec{J}\, d^3x' = 0 \right)$$

Suppose that now I take (5.52) again but use $f = x'_i$ and $g = x'_j$ still with $\nabla' \cdot \vec{J} = 0$:

$$\int \left(f\vec{J} \cdot \nabla' g + g\vec{J} \cdot \nabla' f \right) d^3x' = 0$$

$$\int \left[x'_i\, \vec{J}(\vec{x}\,') \cdot \underbrace{\nabla' x'_j}_{\vec{e}_j} + x'_j\, \vec{J}(\vec{x}\,') \cdot \underbrace{\nabla' x'_i}_{\vec{e}_i} \right] d^3x' = 0$$

$$= \int \left[x'_i\, J_j(\vec{x}\,') + x'_j\, J_i(\vec{x}\,') \right] d^3x' = 0 \quad \text{(A)} \quad (\text{see page } 185)$$

The integral that appears in the second term of (5.52) is:

$$\vec{X}\cdot\int J_i(\vec{x}\,')\,\vec{x}\,'\,d^3x' = \sum_j X_j \int x'_j J_i \, d^3x' =$$

$$= \frac{1}{2}\Big[\underbrace{\sum_j x_j \int x'_j J_i \, d^3x'}_{-\sum_j x_j \int x'_i J_j \, d^3x'} + \sum_j x_j \int x'_j J_i \, d^3x'\Big]$$

using Eq (A) of previous page

$$= -\frac{1}{2}\sum_j x_j \int \underbrace{\left(x'_i J_j - x'_j J_i\right)}_{\left(\vec{x}\,'\times\vec{J}\right)_{k\neq i,j}} d^3x'$$

Note that k is $\neq (i,j)$
But $i\neq j$ as well otherwise the integral cancels. So i, j, k are all <u>different</u>.

$$= -\frac{1}{2}\sum_j\sum_k \underbrace{\epsilon_{ijk}}_{\epsilon_{123}=1,\ \epsilon_{132}=-1 \text{ etc.}} x_j \int \left(\vec{x}\,'\times\vec{J}\right)_k d^3x'$$

$$= -\frac{1}{2}\left[\vec{X}\times\int\left(\vec{x}\,'\times\vec{J}\right)d^3x'\right]_{\text{component } i} = \vec{X}\cdot\int J_i(\vec{x}\,')\,\vec{x}\,'\,d^3x'$$

in general
$$\sum_{j,k}\epsilon_{ijk}\,a_j\,b_k = \left(\vec{a}\times\vec{b}\right)\Big|_i$$

Defining the magnetic moment density or magnetization we get:

$$\vec{\mathcal{M}}(\vec{x}) = \frac{1}{2}\left[\vec{x} \times \vec{J}(\vec{x})\right]$$

and the integral is the magnetic moment $\vec{m}$:

$$\boxed{\vec{m} = \frac{1}{2}\int\left[\vec{x}' \times J(\vec{x}')\right] d^3x'} \qquad (5.54)$$

$\vec{m} \times \vec{x} = -(\vec{x} \times \vec{m})$

Then $$\boxed{\vec{A}(\vec{x}) \approx \frac{\mu_0}{4\pi} \cdot \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}} \qquad (5.55)$$

$\longrightarrow$ similar to (4.10) $\frac{1}{4\pi\epsilon_0} \cdot \frac{\vec{p} \cdot \vec{x}}{r^3}$

Lowest nonvanishing term of a localized steady current distribution

so that $\int \nabla' \cdot (\ldots) = 0$

i.e. $\nabla \cdot \vec{J} = 0$

Now, use $\vec{B} = \nabla \times \vec{A}$

$$\boxed{\vec{B}(\vec{x})} = \nabla \times \left[\frac{\mu_0}{4\pi} \cdot \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3} \right] =$$

$\vec{n}$ is unit vector in the direction $\vec{x}$.

$$= \boxed{\frac{\mu_0}{4\pi} \frac{\left(3\vec{n}(\vec{n}\cdot\vec{m}) - \vec{m}\right)}{|\vec{x}|^3}} \qquad (5.56)$$

Proof

In general $\left.\nabla \times (\vec{a} \times \vec{b})\right|_i = \sum_{jk} \epsilon_{ijk} \nabla_j \underbrace{(\vec{a}\times\vec{b})_k}_{\sum_{lm} \epsilon_{klm} a_l b_m} =$

$$= \sum_{jklm} \epsilon_{ijk} \epsilon_{klm} \nabla_j a_l b_m$$

Consider now $\vec{a} = \vec{m}$ which is $\vec{x}$-independent.
Then $\nabla_j a_l = 0$. Take $\vec{b} = \vec{x}/|\vec{x}|^3$ also.

Thus, so for $\left.\nabla \times \left(\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3}\right)\right|_i = \sum_{jklm} \epsilon_{ijk}\epsilon_{klm} m_l \underbrace{\nabla_j \left(\frac{x_m}{|\vec{x}|^3}\right)}$

$$\to \frac{1}{|\vec{x}|^3} \underbrace{\nabla_j x_m}_{\delta_{jm}} + x_m \underbrace{\nabla_j \left(\frac{1}{|\vec{x}|^3}\right)}$$

$$\nabla_j \frac{1}{(x_1^2+x_2^2+x_3^2)^{3/2}} = -\frac{3}{\cancel{2}} \frac{\cancel{2} x_j}{(x_1^2+x_2^2+x_3^2)^{5/2}}$$

$$= -\frac{3x_j}{|\vec{x}|^{5/2}}$$

Then $\nabla_j \dfrac{x_m}{|\vec{x}|^3} = \dfrac{\delta_{jm}}{|\vec{x}|^3} - \dfrac{3x_j x_m}{|\vec{x}|^{5/2}}$

$$= \frac{\delta_{jm}}{|\vec{x}|^3} - \frac{3n_j n_m}{|\vec{x}|^3}$$

since $\vec{x} = \vec{n}\,|\vec{x}|$, then $\vec{n} = \dfrac{\vec{x}}{|\vec{x}|}$

$n_j = \dfrac{x_j}{|\vec{x}|}$

We get so far:

$$\left.\nabla \times \left(\frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}\right)\right|_i = \sum_{jklm} \epsilon_{ijk}\epsilon_{klm} m_l \frac{(\delta_{jm} - 3n_j n_m)}{|\vec{x}|^3} =$$

$$= \frac{1}{|\vec{x}|^3}\left[\underbrace{\sum_{jklm} \epsilon_{ijk}\epsilon_{klm} m_l \delta_{jm}}_{\sum_{jkl} \epsilon_{ijk}\epsilon_{klj} m_l \,=} - 3\sum_{jklm} \epsilon_{ijk}\epsilon_{klm} m_l n_j n_m\right]$$

$$= +\sum_l \underbrace{\left(\sum_{jk} \epsilon_{ijk}\epsilon_{ljk}\right)}_{} m_l$$

If $i \neq l$, then (e.g. 1 and 2) then always one of the ϵ symbols is 0.

If $i = l$ then we have. We get $(\ldots) \, 2\delta_{il}$

example $i = l = 1$

$\sum_{jk} \epsilon_{1jk}\epsilon_{1jk} = \epsilon_{123}\epsilon_{123} + \epsilon_{132}\epsilon_{132} = 2$

$$i=1: \sum_{jklm} \epsilon_{1jk}\epsilon_{klm} m_l n_j n_m =$$

$$= \overbrace{\epsilon_{123}}^{1} \epsilon_{3lm} m_l n_2 n_m$$

$$+ \underbrace{\epsilon_{132}}_{-1} \epsilon_{2lm} m_l n_3 n_m =$$

$$= n_2 \left(\overbrace{\epsilon_{312}}^{+1} m_1 n_2 + \underbrace{\epsilon_{321}}_{-1} m_2 n_1 \right)$$

$$- n_3 \left(\underbrace{\epsilon_{213}}_{-1} m_1 n_3 + \underbrace{\epsilon_{231}}_{+1} m_3 n_1 \right) =$$

$$= m_1 n_2^2 + m_1 n_3^2 - m_2 n_1 n_2 - m_3 n_3 n_1$$
$$+ m_1 n_1^2 - m_1 n_1^2$$

$$= m_1 \underbrace{(n_1^2 + n_2^2 + n_3^2)}_{=1} - n_1 [m_2 n_2 + m_3 n_3 + m_1 n_1]$$

$$= m_1 - n_1 (\vec{m} \cdot \vec{n}) \quad \text{(same for } i=2 \text{ and } i=3)$$

Overall:

$$\nabla \times \left(\frac{\vec{m} \times \vec{x}}{|\vec{x}|^3} \right) = \frac{1}{|\vec{x}|^3} \left[2\vec{m} - 3\left(\vec{m} - \vec{n}(\vec{m} \cdot \vec{n}) \right) \right]$$

$$= \frac{1}{|\vec{x}|^3} \left[3\vec{n}(\vec{m} \cdot \vec{n}) - \vec{m} \right]$$

Q.E.D.

If $\vec{m} = m\hat{e}_z$ and $\vec{n} = \hat{e}_z$

then $\vec{n}(\vec{n}.\vec{m}) = m\hat{e}_z$

and $3\vec{n}(\vec{n}.\vec{m}) - \vec{m} = 2m\hat{e}_z$

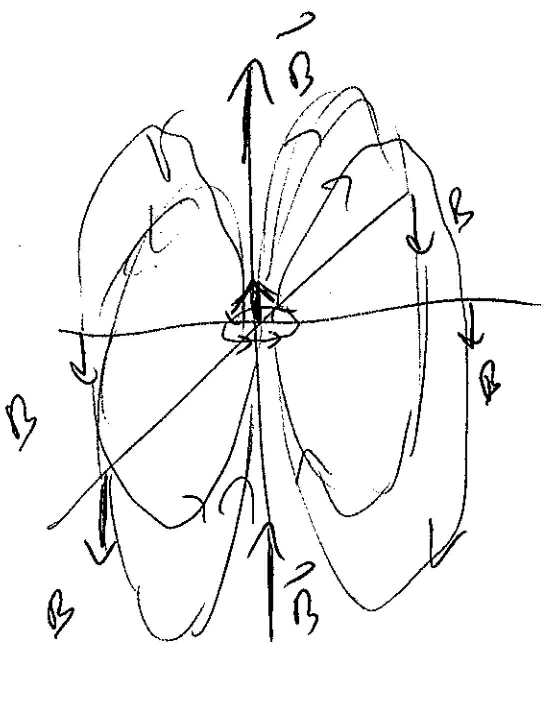
If $\hat{n} = -\hat{e}_z$ (along $-z$ axis)

$\vec{n}.\vec{m} = -m\hat{e}_z$

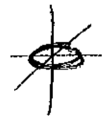
but $\vec{n} = -e_z$, thus the result is the same as for $+z$ axis.

If $\hat{n} = \hat{e}_x$ (along x axis), then

$\vec{n}.\vec{m} = 0$ and $\vec{B}$ points along $-\vec{m} = -m\hat{e}_z$

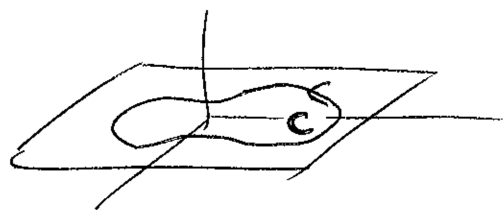


like an electric dipole

A special case of this result is the case of a circular loop  that we analyzed before.

> Far away from any localized current distribution the magnetic field is that of a magnetic dipole with $\vec{m} = \frac{1}{2}\int [\vec{x}' \times \vec{J}(\vec{x}')]\, d^3\vec{x}'$.

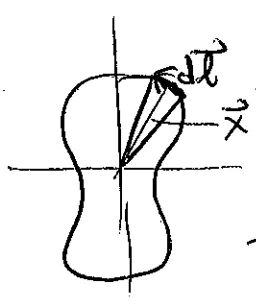
For a current confined to a plane, with a loop of arbitrary form



then $$\vec{m} = \frac{1}{2}\oint_C (\vec{x}' \times I\, d\vec{l})$$

$C \leftarrow \vec{J}(\vec{x}')$ is only $\neq 0$ along the curve C. Thus $\int d^3x' \to \oint_C d\vec{l}$

In addition, seen from above:



$\frac{1}{2}|\vec{x} \times d\vec{l}|$ = elementary area of a small triangle (not rectangle; we need the 1/2)

Then $\frac{1}{2}|\vec{x} \times d\vec{l}| = da$

and $$\boxed{\vec{m} = I \times \text{Area}_{\text{loop}}} \quad (5.57)$$

$\uparrow$ (5.42) is a special case.