5.8 Macroscopic Equations, Boundary Conditions

We want to connect the "micro" with the "macro" descriptions.

First, note that $\nabla \cdot \vec{B} = 0$ is always valid, even at microscopic level. Then, at macro level too:

$$\boxed{\nabla \cdot \vec{B} = 0}$$

This means we can talk about a vector potential $\vec{A}(\vec{x})$ even in the presence of matter.

The many molecular magnetic moments $\vec{m}_i$ produce in average a macroscopic magnetization

$$\vec{M}(\vec{x}) = \sum_i \underbrace{N_i}_{\text{\# of molecules in small volume}} \underbrace{\langle m_i \rangle}_{\text{average of } \vec{m}_i \text{ per small volume at point } \vec{x}}$$

small volume: something like a "unit cell"

This magnetic moment can have any origin, including the intrinsic spin of the electrons.

Of course, we can also have in the system a macroscopic current density $\vec{J}(\vec{x})$.

Overall: keeping only the magnetic dipole moment

Similar to (4.30) but for currents

$$\vec{A}(\vec{x}) \doteq \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \underbrace{\frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}_{\text{This is using (5.55) and shifting coordinates.}}\right]d^3x'$$

Like done before when deriving (4.31), particularly using page 29 (bottom Eq.) we get:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \left(\vec{M}(\vec{x}')\times\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right)\right]d^3x'$$

Now we use a vector identity:

$$\boxed{\nabla\times(\psi\vec{A}) = \psi(\nabla\times\vec{A}) + \underbrace{(\nabla\psi)\times\vec{A}}_{-\vec{A}\times(\nabla\psi)}}$$

Consider $\psi = \frac{1}{|\vec{x}-\vec{x}'|}$, $\vec{A} = \vec{M}(\vec{x}')$. Then:

$$\vec{M}\times\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\times\vec{M}$$

$$= -\nabla'\times\left(\frac{\vec{M}}{|\vec{x}-\vec{x}'|}\right) + \frac{1}{|\vec{x}-\vec{x}'|}\left(\nabla'\times\vec{M}(\vec{x}')\right)$$

The integral of the first term cancels

$$\int \nabla \times \left(\frac{\vec{M}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} \right) d^3x'$$ since it reduces to a surface integral which vanishes if all functions are "well behaved" and localized.

Stokes theorem

$$\int_V \nabla \times \vec{F} \cdot d\vec{S} = \oint_S \vec{F} \cdot d\vec{s}$$

then:

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \left(\frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} + \frac{[\nabla' \times \vec{M}(\vec{x}\,')]}{|\vec{x}-\vec{x}\,'|} \right) d^3x'} \quad (5.78)$$

Thus, it is as if the macroscopic medium provides an effective current density that includes $\underline{\nabla' \times M(\vec{x}\,')}$.

If $\vec{A}$ is like (5.32), which was derived in free space, by just replacing $\vec{J}$ by $\vec{J}_{eff}$, then (5.26) will also hold if $\vec{J} \to \vec{J}_{eff}$.

Then:

$$\boxed{\nabla \times \vec{B} = \mu_0 \underbrace{\left[\vec{J} + \nabla \times \vec{M}\right]}_{\vec{J}_{eff}}} \quad (5.80)$$

If $\vec{B}$ and $\vec{M}$ are combined as

$$\boxed{\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}}$$

← analogous to $\vec{D}$ in electrostatics

then

$$\nabla \times \vec{H} = \nabla \times \left(\frac{\vec{B}}{\mu_0}\right) - \nabla \times \vec{M} \overset{(5.80)}{=} \vec{J}$$

↑ not effective but "external" or "free"

(depends on $\vec{J}$ sources real)

$$\boxed{\nabla \times \vec{H} = \vec{J}} \quad (5.82)$$

and

$$\boxed{\nabla \cdot \vec{B} = 0}$$

names:
B = magnetic induction
H = magnetic field

As in the case of electrostatics, to complete the set of formulas for calculations we need a relation between $\vec{H}$ and $\vec{B}$. Assuming <u>linearity</u> and <u>isotropy</u>

$$\boxed{\vec{B} = \mu \vec{H}}$$

μ = magnetic permeability

The boundary conditions at an interface are:

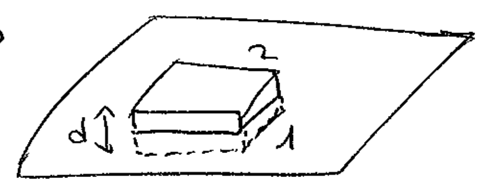
From $\nabla \cdot \vec{B} = 0$, the integral over a ~~small~~ volume V

$$\underbrace{\int_V \nabla \cdot \vec{B}\, d^3x'}\ = 0 \quad \text{of course.}$$

But this is $\oint_{\text{surface}} \vec{B} \cdot \vec{n}\, da$ due to ~~[illegible]~~ divergence theorem.

Then: $$\oint_{\text{surface}} \vec{B} \cdot \vec{n}\, da = 0$$

Now, apply this to a small volume at the interface as

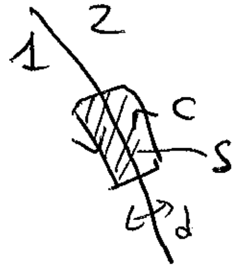


As $d \to 0$, the only integrals that matter are the top and bottom ones. They become:

$$\boxed{(\vec{B}_2 - \vec{B}_1) \cdot \vec{n} = 0} \qquad (5.86)$$

(See I.5 for details)

Now consider a small loop as in Fig I.4

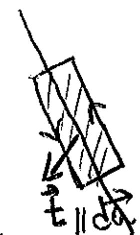


We deduced

$$\nabla \times \vec{H} = \vec{J} \qquad (S.82)$$

Apply now Stokes theorem

$$\int_S (\nabla \times \vec{F}) \cdot d\vec{a} = \int_{\partial S} \vec{F} \cdot d\vec{s}$$



$$\int_S \nabla \times \vec{H} \cdot d\vec{a} = \underbrace{\oint_C \vec{H} \cdot d\vec{\ell}}_{\text{"side" contributions vanish as "d"} \to 0} = \int_S \vec{J} \cdot d\vec{a} = K \cdot t$$

"Surface current" $\vec{K}$

t = unit vector tangencial to surface (points in same direction as $d\vec{a}$)

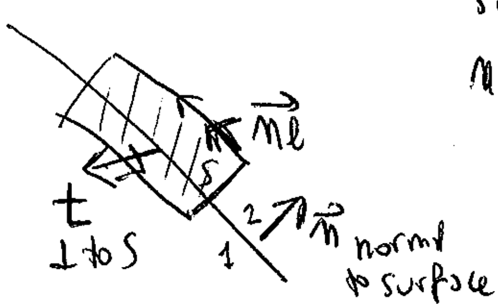
(Units of K) are (units of J) x area.

Then, we get:

$$(\vec{H}_2 - \vec{H}_1) \cdot \underbrace{\vec{n}_\ell}_{} = \vec{K} \cdot \vec{t}$$

unit vector ~~shon~~ as shown. Note that

$$\vec{n}_\ell = \vec{t} \times n$$



$$\underbrace{(\vec{H}_2 - \vec{H}_1) \cdot (\vec{t} \times \vec{n})}_{\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})} = \vec{K} \cdot \vec{t} = \vec{t} \cdot \vec{K}$$

Then

$$\vec{t} \cdot \left[\vec{n} \times \left(\vec{H}_2 - \vec{H}_1\right)\right] = \vec{t} \cdot \vec{K}$$

"$\vec{b}$" "$\vec{c}$" "$\vec{a}$"

Then

$$\boxed{\vec{n} \times \left(\vec{H}_2 - \vec{H}_1\right) = \vec{K}} \qquad (5.87)$$

$\vec{n}$ points from 1 to 2 i.e. it is $\vec{n}_{12}$.

Alternatively, using $\vec{B} = \mu \vec{H}$,

$\vec{B}_2 \cdot \vec{n} = \vec{B}_1 \cdot \vec{n}$ becomes $\boxed{\vec{H}_2 \cdot \vec{n} = \frac{\mu_1}{\mu_2} \vec{H}_1 \cdot \vec{n}}$

and $\boxed{\vec{n} \times \left(\frac{\vec{B}_2}{\mu_2} - \frac{\vec{B}_1}{\mu_1}\right) = \vec{K}}$

5.9 How do we solve these problems?

Basically we need to solve

$$\nabla \cdot \vec{B} = 0 \ , \quad \nabla \times \vec{H} = \vec{J}$$

with some relation between $\vec{B}$ and $\vec{H}$; $\vec{H} = \vec{H}(\vec{B})$

(A). Always we can say $\vec{B} = \nabla \times \vec{A}$

plus $\quad \nabla \times \vec{H}[\nabla \times \vec{A}] = \vec{J}$

and these are then equations in $\vec{A}$ only.

For linear media, $\vec{B} = \mu \vec{H}$, and then

$$\nabla \times \frac{(\nabla \times \vec{A})}{\mu} = \vec{J}$$

μ could be region 1 | region 2

If μ is constant in some region, then in that region

$$\frac{1}{\mu} \nabla \times (\nabla \times \vec{A}) = \frac{1}{\mu}\left(\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}\right) = \vec{J}$$

Choosing $\nabla \cdot \vec{A} = 0$, then $\quad -\nabla^2 \vec{A} = \mu \vec{J}$

Of course, in addition we have to deal with boundary conditions.

(B). $\vec{J}=\vec{0}$; Magnetic Scalar potential

If in some region $\vec{J}=\vec{0}$, then $\nabla\times\vec{H}=\vec{0}$ and $\vec{H}$ can be written as $\vec{H}=-\nabla\Phi_M$ (similarly as $\vec{E}=-\nabla\phi$)

If the medium is linear then:

$$\nabla\cdot\vec{B}=\nabla\cdot(\mu\vec{H})=-\nabla\cdot(\mu\nabla\Phi_M)=0$$

If μ is "piecewise constant", then

$$\nabla^2\Phi_M=0 \quad \text{in each region}$$

[Typical example: medium in uniform external magnetic field.]

(C) Hard Ferromagnets

Here a $\vec{M}$ that is considered fixed, i.e. <u>independent of applied fields</u>. We will also assume $\vec{J}=0$.

(a) We will use (B) above since $\vec{J}=0$.

$$\nabla\cdot\vec{B}\underset{\uparrow}{=}\nabla\cdot(\mu_0(\vec{H}+\vec{M}))=0$$

(5.81)

In addition (B) says $\vec{H} \underset{\uparrow \text{ if no currents}}{=} -\nabla \Phi_M$. Then

$$\nabla \cdot \left(\mu_0 \left(-\nabla \Phi_M\right) + \vec{M}\right) = 0$$

~~assumed piecewise~~ μ_0 always constant, thus I drop it.

$$\cancel{\mu_0}\left[-\nabla^2 \Phi_M\right] + \nabla \cdot \vec{M} = 0$$

$$+\nabla^2 \Phi_M = -\rho_M \underset{\text{def.}}{=} \frac{+\nabla \cdot \vec{M}}{\cancel{\mu_0}}$$

From previous experience in electrostatics:

$$\Phi_M = -\frac{1}{4\pi} \int \frac{\left[\nabla' \cdot M(\vec{x}')\right]}{|\vec{x} - \vec{x}'|} \, d^3x'$$

since

$$\nabla^2 \Phi_M = -\frac{1}{4\pi} \int \left[\nabla' \cdot M(\vec{x}')\right] \underbrace{\nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}'|}\right)}_{-4\pi \delta(\vec{x} - \vec{x}')} d^3x' =$$

$$= \nabla \cdot M(\vec{x}) \checkmark$$

If $\vec{M}$ is "well-behaved and localized", then we can integrate by parts as done many times before:

$$\nabla' \cdot (f\vec{A}) = \nabla' f \cdot \vec{A} + f(\nabla' \cdot \vec{A})$$

$$\underbrace{\int \nabla' \cdot (f\vec{A})}_{\to 0} = \int \underbrace{(\nabla' f \cdot \vec{A})}_{\vec{A}\cdot\nabla' f} + \int f(\nabla' \cdot \vec{A})$$

Thus: $\displaystyle\int f(\nabla' \cdot \vec{A}) = -\int \vec{A} \cdot \nabla' f$

Then, using $\vec{A} = \vec{M}$ and $f = \dfrac{1}{|\vec{x}-\vec{x}'|}$

$$\boxed{\Phi_M = +\frac{1}{4\pi}\int \vec{M}(\vec{x}') \cdot \underbrace{\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)}_{-\nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)} d^3x' = -\frac{1}{4\pi}\nabla \cdot \int \frac{\vec{M}(\vec{x}')\, d^3x'}{|\vec{x}-\vec{x}'|}} \quad (5.98)$$

If there are $\underline{\underline{\text{no}}}$ boundary surfaces!

Let us now assume "hard" ferromagnets: $M(\vec{x})$ inside a volume V and 0 outside.

Then, in this case there is a magnetic surface charge density "σ_M". Then, the solution will be the one before plus the analog of (1.23) for electrostatics. We need to find the value of σ_M. The value can be obtained similarly as the derivation of (4.46)

Remember $\rho_M = -\nabla . M$ discontinuous thus $\frac{\partial M}{\partial n} =$ surface

$$\sigma_M = \vec{n} . \vec{M}$$

$\vec{n}$ $\vec{M}$

$$\sigma = -(\vec{P}_2 - \vec{P}_1) . \vec{n}_{21}$$

Use $\vec{P}_2 = 0$

$\vec{P}_1 = \vec{M}$

$\vec{n}_{21} = \vec{n}$

Then:

$$\boxed{\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int_V \frac{[\nabla' . M(\vec{x}')]}{|\vec{x}-\vec{x}'|} d^3x' + \frac{1}{4\pi}\oint_S \frac{\vec{n}' . M(\vec{x}')}{|\vec{x}-\vec{x}'|} da'} \qquad (5.100)$$

We leave method C. (b) "Vector potential" for the readers to discuss.
(students)