

## 6.1 Maxwell Equations

) Thus far the basic equations we have used are:

Coulomb

$$\nabla \cdot \vec{D} = \rho \quad \leftarrow \text{Chapter 4}$$

Ampere

$$\nabla \times \vec{H} = \vec{J} \quad (\nabla \cdot \vec{J} = 0) \quad \leftarrow \begin{matrix} (5.26) \\ \text{or } (5.70) \end{matrix}$$

Faraday

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \leftarrow (5.143)$$

no mag. monopoles

$$\nabla \cdot \vec{B} = 0$$

These are not yet Maxwell's equations. Note that with the exception of the third (Faraday's), the others were derived for steady-state cases. Then, perhaps they will change if the phenomena becomes time dependent.

In general the continuity equation for charge and current says:

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\text{But } \vec{J} = \nabla \cdot \vec{D}$$

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left( \frac{\partial \vec{D}}{\partial t} \right) \quad \text{and this becomes:}$$

$$\boxed{\nabla \cdot \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = 0}$$

Then, the combination  $\vec{J} + \frac{\partial \vec{D}}{\partial t}$  appears to play an important role. How about replacing  $\vec{J}$  by  $\vec{J} + \frac{\partial \vec{D}}{\partial t}$  in Ampere's law?

It becomes

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

① For time-indep. phenomena,  $\frac{\partial \vec{D}}{\partial t} = 0$  and we go back to the usual  $\nabla \times \vec{H} = \vec{J}$ .

② It is consistent with the continuity equation because

$$\underbrace{\nabla \cdot (\nabla \times \vec{H})}_{=0} = \nabla \cdot \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = 0$$

So it is a good guess! And also it establishes that changing an electric field in time generates a magnetic field, the analog of Faraday's law (changing  $\vec{B}$  generates an  $\vec{E}$ ). This new law is correct: it predicts electromagnetic radiation; a moving charge produces an e.m. field. This was a prediction in Maxwell's time that has been much verified.

The final set of equations is

$\nabla \cdot \vec{D} = \rho$	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$
$\nabla \cdot \vec{B} = 0$	$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

(6.6)

units: SI as before

The other two equations will lead to the second-order diff. eqs. that we must solve.

Let us consider first the case of the vacuum:

In this case (vacuum)  $\vec{D} = \epsilon_0 \vec{E}$ ,  $\vec{B} = \mu_0 \vec{H}$ .

Then:

$$\nabla \cdot \vec{D} = \rho$$

$$\epsilon_0 \nabla \cdot \vec{E} = \rho$$

$$\nabla \cdot \left( -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho}{\epsilon_0}$$

$$\boxed{\nabla^2 \Phi + \frac{\partial}{\partial t} (\nabla \cdot \vec{A}) = -\frac{\rho}{\epsilon_0}} \quad (6.10)$$

and  $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

$$\mu_0^{-1} (\nabla \times \vec{B}) = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\frac{1}{\mu_0} (\nabla \times (\nabla \times \vec{A})) = \vec{J} + \epsilon_0 \frac{\partial}{\partial t} \left( -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \right)$$

Here we use  $\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

$$\nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \left[ -\nabla \frac{\partial \Phi}{\partial t} - \frac{\partial^2 \vec{A}}{\partial t^2} \right]$$

## 6.2 Vector and Scalar Potentials

The Maxwell equations relate the electric and magnetic fields (two diff. eqs. of first order). They could be solved as they are in special cases. However, as we did in electrostatics, it is convenient to introduce "potentials" to reduce the number of diff. eq. (but transform them into second order).

We are already familiar with  $\Phi$  in electrostatics and the vector potential  $\vec{A}$ , but we want to see how to generalize them to the full Maxwell equations.

Since  $\nabla \cdot \vec{B} = 0$ , then  $\boxed{\vec{B} = \nabla \times \vec{A}}$  (6.7)

From  $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

$$\nabla \times \vec{E} + \frac{\partial}{\partial t}(\nabla \times \vec{A}) = 0, \quad \nabla \times \left( \vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

Thus:  $\underbrace{\vec{E} + \frac{\partial \vec{A}}{\partial t}}_{\text{or}} = -\nabla \Phi$  since  $\nabla \times \nabla \Phi = 0$

or  $\boxed{\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}}$  (6.9)

Then, by construction (6.7) and (6.9) "take care" of two of Maxwell's equations.

or

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla \left( \nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \Phi}{\partial t} \right) = -\mu_0 \vec{J}$$

But  $\mu_0 \epsilon_0 = \frac{1}{c^2}$  (see page 3)

$$\boxed{\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla \left( \nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = -\mu_0 \vec{J}} \quad (6.11)$$

Note that (6.10) and (6.11) are coupled  
i.e. both  $\Phi$  and  $\vec{A}$  appear in both equations.

However, there is an extra freedom to exploit  
to get the equations to uncouple:

Since  $\vec{B} = \nabla \times \vec{A}$ , then if I add to  $\vec{A}$   
the gradient of a function  $\Lambda$ , then  $\vec{B}$  is still the same.

Thus,  $\boxed{\vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla \Lambda}$  leaves  $\vec{B}$  unchanged.

But what happens to  $\vec{E}$ ?

$\vec{E} \rightarrow \vec{E}' = -\nabla \Phi - \frac{\partial}{\partial t} (\vec{A} + \nabla \Lambda)$ . To compensate  
 $\Phi$  should change as  $\boxed{\Phi \rightarrow \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}}$

$$-\nabla \Phi \rightarrow -\nabla \Phi' = -\nabla \Phi + \frac{\partial}{\partial t} (\nabla \Lambda)$$

and this leaves  $\vec{E}$  unchanged as well.

This means that we have a "freedom":  
we can choose  $(\vec{A}, \Phi)$  such that

$$\boxed{\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0} \quad (6.14)$$

"Lorentz condition"

With this extra request

$$\nabla^2 \Phi + \underbrace{\frac{\partial}{\partial t} (\nabla \cdot \vec{A})}_{-\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}} = -\frac{f}{\epsilon_0} \quad \text{becomes}$$

$$\boxed{\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{f}{\epsilon_0}} \quad (6.15)$$

and (6.11) becomes

$$\boxed{\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}} \quad (6.16)$$

So the combination (6.14), (6.15), (6.16)  
are equivalent to the Maxwell equations (in vacuum).

### 6.3 Gauge transformations

$$\left. \begin{array}{l} \vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla \Lambda \\ \Phi \rightarrow \Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \end{array} \right\} \begin{array}{l} \text{are called} \\ \text{"gauge transformations"} \end{array}$$

and the invariance of  $\vec{E}$  and  $\vec{B}$  under this operation is called "gauge invariance".

Note that if  $\vec{A}$  and  $\Phi$  satisfy (6.10, 6.11) but not the Lorenz condition, then by a gauge transformation we can make them satisfy that condition.

$$\nabla \cdot \vec{A}' + \frac{1}{c^2} \frac{\partial \Phi'}{\partial t} = 0 = \nabla \cdot (\vec{A} + \nabla \Lambda) + \frac{1}{c^2} \frac{\partial}{\partial t} \left( \Phi - \frac{\partial \Lambda}{\partial t} \right)$$

we request  
 this of the  
 gauge transformed  
 of  $\vec{A}$  and  $\Phi$ .

$$= \nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} + \nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2}.$$

Thus, if we find a function  $\Lambda$  such that

$$\boxed{\nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = - \left( \nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right)} \quad (6.18)$$

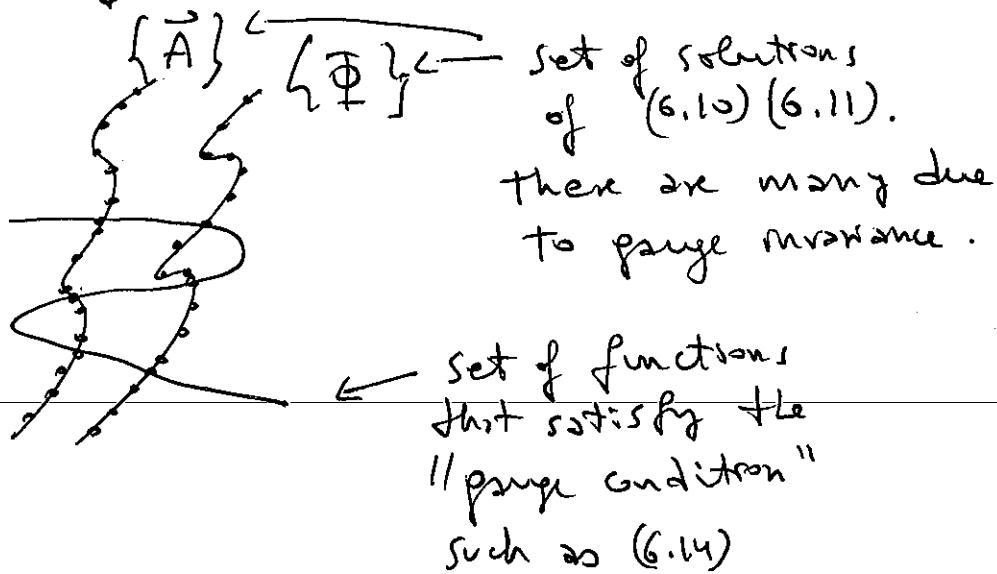
then, the transformed  $\vec{A}'$  and  $\Phi'$  will satisfy Eqs. (6.14, 6.15, 6.16)

Even if  $\vec{A}$  and  $\Phi$  already satisfy the Lorenz condition, there are other  $\vec{A}'$  and  $\Phi'$  that also do. Repeating the previous math, if  $\vec{A}$  and  $\Phi$  already satisfy  $\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0$ ,

then

$$\nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = 0 \quad (6.20)$$

is sufficient to preserve the Lorenz condition. If (6.20) has "many solutions," then "there are many"  $\vec{A}$ 's and  $\Phi$ 's that satisfy the Lorenz condition.



The intersection satisfies both Eqs. (6.10) (6.11) and the gauge condition (6.14). Note that (6.20) implies that the intersection can be "more than one point" (i.e. more than one function).

Finding another gauge is like finding another "line" intersecting the set of  $(\vec{A}, \Phi)$  solutions of Eqs. (6.10) (6.11).

One example is the Coulomb gauge

$$\boxed{\nabla \cdot \vec{A} = 0}$$

From (6.10), we get:

$$\nabla^2 \Phi = -\rho/\epsilon_0 \quad (6.22)$$

(the usual Poisson Eq.)

and the solution is

$$\boxed{\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|} d^3x'} \quad (6.23)$$

Note that compared with electrostatics now  $\rho$  can have a  $t$  dependence. In this case  $\Phi$  looks like the "instantaneous Coulomb potential" due to  $\rho(\vec{x}, t)$ .

The other Eq. (6.11) becomes:

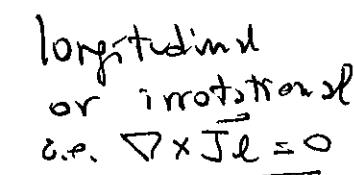
$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \frac{1}{c^2} \nabla \left( \frac{\partial \Phi}{\partial t} \right) \quad (6.24)$$

Note that the last term  $\frac{1}{c^2} \nabla \cdot \frac{\partial \vec{\Phi}}{\partial t}$

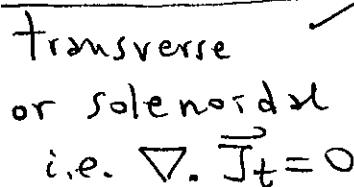
) has  $\nabla \times \frac{1}{c^2} \nabla \left( \frac{\partial \vec{\Phi}}{\partial t} \right) = 0$

and for this reason it is called "irrotational" which means vanishing curl.

The current  $\vec{J}$  can be written  $\vec{J} = \vec{J}_l + \vec{J}_t$



longitudinal  
or irrotational  
i.e.  $\nabla \times \vec{J}_l = 0$



transverse  
or solenoidal  
i.e.  $\nabla \cdot \vec{J}_t = 0$

Given a current  $\vec{J}$ , we can construct  $\vec{J}_t$  and  $\vec{J}_l$  as follows:

$$\boxed{\vec{J}_l = -\frac{1}{4\pi} \nabla \left[ \int \frac{\nabla \cdot \vec{J}}{|x-x'|} d^3 x' \right] \text{ no clearly } \nabla \times \vec{J}_l = 0 \text{ arbitrary since } \nabla \times \nabla \psi = 0}$$

$$\vec{J}_t = \frac{1}{4\pi} \nabla \times \left[ \nabla \times \left[ \int \frac{\vec{J}}{|x-x'|} d^3 x' \right] \right]$$

In the formula for  $\vec{J}_t$  I will use

$$\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V} \quad (\vec{V} = \text{arbitrary})$$

$$\vec{J}_t = \frac{1}{4\pi} \left[ \nabla \left( \nabla \cdot \int \frac{\vec{J} d^3x'}{|\vec{x} - \vec{x}'|} \right) - \underbrace{\nabla \cdot \nabla \int}_{\nabla^2} \frac{\vec{J} d^3x'}{|\vec{x} - \vec{x}'|} \right]$$

acts only on  $\frac{1}{|\vec{x} - \vec{x}'|}$

since  $\vec{J}$  has a  $\vec{x}'$  dependence

$$-\nabla^2 \int \frac{\vec{J} d^3x'}{|\vec{x} - \vec{x}'|} = - \int \vec{J} d^3x' \left[ 4\pi \delta(\vec{x} - \vec{x}') \right] =$$

$$\left( \nabla^2 \frac{1}{|\vec{x} - \vec{x}'|} \right) = -4\pi \delta(\vec{x} - \vec{x}')$$

$$= 4\pi \vec{J}$$

$$\vec{J}_t = \frac{1}{4\pi} \nabla \left( \nabla \cdot \int \frac{\vec{J} d^3x'}{|\vec{x} - \vec{x}'|} \right) + \vec{J}$$

is this  $-J_t$ ?

To prove this use:

$$\left\{ \nabla' \cdot \left( \frac{\vec{J}}{|\vec{x} - \vec{x}'|} \right) d^3x' \right\} = \left\{ (\nabla' \cdot \vec{J}) \frac{1}{|\vec{x} - \vec{x}'|} d^3x' + \left\{ \nabla' \frac{1}{|\vec{x} - \vec{x}'|} \right\} \cdot \vec{J} d^3x' \right\}$$

$$- \nabla \left( \frac{1}{|\vec{x} - \vec{x}'|} \right)$$

$$\left\{ \int \frac{\vec{J}}{|\vec{x} - \vec{x}'|} \cdot \hat{n} da \right\}$$

$= 0$   
if  $\vec{J} \rightarrow 0$  at infinite i.e. bounded currents

Then:

$$\int \left( \nabla^! \cdot \vec{J} \right) \frac{\perp}{|\vec{x} - \vec{x}'|} d^3x' = \int \nabla \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) \cdot \vec{J} d^3x'$$

$$= \nabla \cdot \int \frac{\vec{J}}{|\vec{x} - \vec{x}'|} d^3x'$$

Thus:

$$\underbrace{\frac{1}{4\pi} \nabla \left( \nabla \cdot \int \frac{\vec{J}}{|\vec{x} - \vec{x}'|} d^3x' \right)}_{\text{term that appears in } \vec{J}_t \text{ formula.}} = \underbrace{\frac{1}{4\pi} \nabla \left( \int \frac{(\nabla^! \cdot \vec{J})}{|\vec{x} - \vec{x}'|} d^3x' \right)}_{-\vec{J}_e \quad (6.27)}$$

Finally:

$$\vec{J}_t = -\vec{J}_e + \vec{J}$$

or  $\vec{J} = \vec{J}_e + \vec{J}_t$

$\uparrow$                    $\uparrow$   
 (6.27)              (6.28)

Then, the solutions proposed (6.27), (6.28) are correct.

Continuity equation says

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Since  $\Phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|} d^3x'$  (6.23)

$$\frac{\partial \Phi}{\partial t} = \frac{1}{4\pi\epsilon_0} \int \frac{\partial \rho / \partial t}{|\vec{x} - \vec{x}'|} d^3x' = -\frac{1}{4\pi\epsilon_0} \int \frac{(\nabla \cdot \vec{J})}{|\vec{x} - \vec{x}'|} d^3x'$$

$$\boxed{\frac{1}{c^2} \nabla \frac{\partial \Phi}{\partial t}} = -\frac{1}{4\pi\epsilon_0 c^2} \nabla \int \frac{(\nabla \cdot \vec{J})}{|\vec{x} - \vec{x}'|} d^3x'$$

(6.27)

$$\stackrel{\uparrow}{\frac{1}{\epsilon_0 c^2} \vec{J}_e} = \boxed{\mu_0 \vec{J}_e} \quad (6.29)$$
$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

Then, going back to (6.24):

$$\boxed{\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \mu_0 \vec{J}_e = -\mu_0 \vec{J}_t} \quad (6.30)$$

This is why the Coulomb gauge is sometimes called the "transverse gauge".

If there are no sources present, then  $\vec{\Phi} = 0$

(since  $\rho(\vec{x}', t) = 0$ )

and  $\vec{E} = - \frac{\partial \vec{A}}{\partial t}$  (from (6.9))

and  $\vec{B} = \nabla \times \vec{A}$  (always valid)