

6.7 Conservation of Energy

For a single charge the "rate of doing work" is $q \vec{v} \cdot \vec{E}$ with \vec{v} the velocity of the charge.

The magnetic field does not contribute because the magnetic force is perpendicular to \vec{v} .

From (1.18) $W = -q \int_A^B \vec{E} \cdot d\vec{l}$. If I write $W = \int_A^B dW$

then $dW = -q \vec{E} \cdot d\vec{l}$

and $\frac{dW}{dt} = -q \vec{E} \cdot \frac{d\vec{l}}{dt} = -q \vec{E} \cdot \vec{v}$

"rate of doing work"

See also page 40, Exd. 1.11

If there are many charges

$$\frac{dW}{dt} = - \sum_i q_i \vec{E}_i \cdot \vec{v}_i$$

to continuous \Rightarrow

$$= - \int_V d^3x \underbrace{q(\vec{x}) \vec{v}(\vec{x})}_{\vec{J}(\vec{x})} \cdot \vec{E}(\vec{x})$$

like \vec{J}_i

Note that (1.18) talks about the work done to move a charge from A to B. Here we talk about the work done by the fields.

Thus, there is a sign change

$$\frac{dW}{dt} = \int_V d^3x \vec{J}(\vec{x}) \cdot \vec{E}(\vec{x})$$

It is a conversion of EMag energy into mechanical or thermal energy

Now let us use one of Maxwell's equations:

$$(6.62) \quad \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$$

$$\int_V \vec{J} \cdot \vec{E} d^3x = \int_V d^3x \left[\vec{E} \cdot (\nabla \times \vec{H}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right]$$

Now use the vector identity

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H})$$

$$\int_V \vec{J} \cdot \vec{E} d^3x = \int_V d^3x \left(-\nabla \cdot (\vec{E} \times \vec{H}) + \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right)$$

$-\frac{\partial \vec{B}}{\partial t}$ from Maxwell's Eqs.
(6.62)

Suppose the medium is linear i.e.

$$\vec{D} = \epsilon \vec{E} \quad \text{and} \quad \vec{B} = \mu \vec{H}$$

Then

$$\frac{\partial}{\partial t} (\vec{E} \cdot \vec{D}) = \underbrace{\frac{\partial \vec{E}}{\partial t} \cdot \vec{D}}_{\frac{1}{\epsilon} \frac{\partial \vec{D}}{\partial t} \cdot \epsilon \vec{E}} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = 2 \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$$

and

$$\frac{\partial}{\partial t} (\vec{B} \cdot \vec{H}) = \underbrace{\frac{\partial \vec{B}}{\partial t} \cdot \vec{H}}_{\mu \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}} + \vec{B} \cdot \frac{\partial \vec{H}}{\partial t} = 2 \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$$

Then:

$$\frac{\partial}{\partial t} \left[\underbrace{\frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H})}_u \right] = \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$$

and

$$\int_V \vec{J} \cdot \vec{E} d^3x = \int_V d^3x \left[\frac{\partial u}{\partial t} + \nabla \cdot (\vec{E} \times \vec{H}) \right]$$

we call this \vec{S} , the
"Poynting vector"

Since the equation is valid for an arbitrary V , then the integrands must be equal:

$$\boxed{-\vec{J} \cdot \vec{E} = \frac{\partial u}{\partial t} + \nabla \cdot \vec{S}} \quad (6.108)$$

$$\boxed{\vec{S} = \vec{E} \times \vec{H}}$$

like a continuity equation.

dimensions $\left(\frac{\text{Energy}}{\text{Area} \cdot \text{time}} \right)$ In (4.89) we read $W = \frac{1}{2} \int \vec{E} \cdot \vec{D} d^3x$

" u " is the total energy density.

which for a linear medium is

$$W = \frac{\epsilon}{2} \int \vec{E} \cdot \vec{E} d^3x$$

which for $\epsilon = \epsilon_0$ we derived

time for (1.54)

With regards to the magnetic component

(S.148) says

$$W = \frac{1}{2} \int \vec{H} \cdot \vec{B} d^3x.$$

Then: u is the sum of both contributions.

Eq (6.108) (or its integral form) expresses the "conservation of energy":

(a) $\int_V d^3x \frac{\partial u}{\partial t} =$ time rate of change of electromagnetic energy in V
 $\frac{\partial}{\partial t} \left[\int_V d^3x u \right]$

(b) $\int_V d^3x \nabla \cdot \vec{S} = \oint_A \vec{S} \cdot \vec{n} da =$ energy flowing out through the surface per unit time
 (\vec{S} has units $\frac{\text{energy}}{\text{time} \cdot \text{area}}$)

(c) $-\int_V \vec{J} \cdot \vec{E} d^3x =$ (Work done by the fields on the sources inside V , per unit time)] as argued next this can be considered \rightarrow de charges de energy of the sources.
 Again, this is valid only if the medium is strictly linear.

$\vec{J} \cdot \vec{E}$ represents a conversion of electromagnetic energy into mechanical or heat energy.

So $\int_V \vec{J} \cdot \vec{E} d^3x$ is energy given to the charges and currents (moving charges)

Thus:

$$\frac{dE_{\text{mechanical}}}{dt} = \int_V \vec{J} \cdot \vec{E} d^3x$$

Total energy of the particles inside V . Can be "kinetic", "thermal", etc.

Let us assume particles do not move "in" and "out" of V .

time rate of change of the energy of the particles inside V .

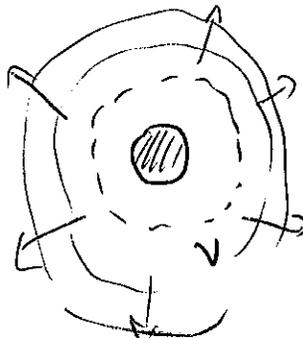
Then, for the combined system (fields + charges):

$$\frac{dE_{\text{tot}}}{dt} = \frac{d}{dt} (E_{\text{mech}} + E_{\text{field}}) = - \int_V d^3x \nabla \cdot \vec{S}$$

$$\int_V \vec{J} \cdot \vec{E} d^3x \rightarrow \frac{d}{dt} \int_V u d^3x$$

$$- \oint_S \vec{n} \cdot \vec{S} da \quad (6.111)$$

Note that in the vacuum:



If spherical waves are taking energy out of V , note that $\vec{S} \cdot \vec{n}$ is (+), thus (-) in front needed to account for energy reduction.

$$E_{\text{field}} = \int_V u \, d^3x = \int_V \frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}) \, d^3x =$$

if in vacuum

$\epsilon_0 \vec{E}$ $\frac{\vec{B}}{\mu_0}$

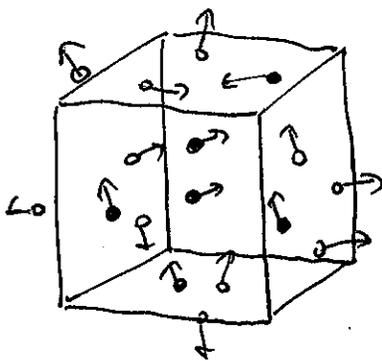
$$= \int_V \frac{1}{2} (\epsilon_0 \vec{E} \cdot \vec{E} + \frac{1}{\mu_0} \vec{B} \cdot \vec{B}) \, d^3x =$$

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

$$= \frac{1}{2} \epsilon_0 \int (\vec{E}^2 + c^2 \vec{B}^2) \, d^3x \quad (6.112)$$

Note: (6.111) $\frac{d}{dt} (E_{\text{mech}} + E_{\text{field}}) = - \oint_S \vec{n} \cdot \vec{S} \, da$

can be visualized in terms of charges and photons:



- = matter. It is assumed not to leave the volume.
- = photons (inside and/or leaving)

$\vec{n} \cdot \vec{S}$ is like the momentum perpendicular to walls of the photons exiting the volume. And energy and mom. are linearly related in EM fields!

The (-) indicates loss of energy if photons leave.

If photons enter then $\vec{n} \cdot \vec{S}$ negative and $-\vec{n} \cdot \vec{S}$ is a gain.