

Consider now the conservation of momentum.

The e.m. force on a particle of charge q is:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

For many particles, and going to a continuum limit via $q \rightarrow p$ and $q\vec{v} \rightarrow \vec{J}$, we obtain

$$\vec{F} = \int \left((\rho(\vec{x}) \vec{E}(\vec{x}) + [\vec{J}(\vec{x}) \times \vec{B}(\vec{x})]) \right) d^3x \quad (6.114)$$

and $\vec{F} = \frac{d\vec{P}_{\text{mech}}}{dt}$. \vec{F} = total Electromagnetic force.
 \vec{P}_{mech} = sum momenta of all the particles.

Consider now vacuum. In this case:

$$(6.62) \rightarrow \nabla \cdot \vec{D} = \rho \text{ becomes } \epsilon_0 \nabla \cdot \vec{E} = \rho$$

$$\text{and } \vec{J} = \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} \text{ becomes } \vec{J} = \nabla \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Replacing into (6.114) :

$$q\vec{E} + (\vec{J} \times \vec{B}) = \epsilon_0 \vec{E}(\nabla \cdot \vec{E}) + \underbrace{\left(\frac{(\nabla \times \vec{B}) \times \vec{B}}{\mu_0} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \times \vec{B} \right)}_{-\vec{B} \times (\nabla \times \vec{B})} - \underbrace{\frac{\vec{B} \times \partial \vec{E}}{\mu_0}}_{-\vec{B} \times \frac{\partial \vec{E}}{\partial t}}$$

$$\text{and } \epsilon_0 \mu_0 = \frac{1}{c^2}$$

$$q\vec{E} + (\vec{J} \times \vec{B}) = \epsilon_0 \left[\vec{E}(\nabla \cdot \vec{E}) + \vec{B} \times \frac{\partial \vec{E}}{\partial t} - c \vec{B} \times (\nabla \times \vec{B}) \right] \quad (6.115)$$

Writing

$$\vec{B} \times \frac{\partial \vec{E}}{\partial t} = \frac{\partial}{\partial t} (\vec{B} \times \vec{E}) - \frac{\partial \vec{B}}{\partial t} \times \vec{E}$$

$$= - \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) + \vec{E} \times \underbrace{\frac{\partial \vec{B}}{\partial t}}_{-\nabla \times \vec{E}}$$

and adding $c^2 \vec{B} (\nabla \cdot \vec{B}) = 0$, we get: (6.62)

$\left. \begin{array}{l} \text{since } \nabla \cdot \vec{B} = 0 \\ \text{in the Max. Sys.} \end{array} \right\}$

$$\rho \vec{E} + \vec{j} \times \vec{B} = \epsilon_0 \left[\vec{E} (\nabla \cdot \vec{E}) + c^2 \vec{B} (\nabla \cdot \vec{B}) \right. \\ \left. - \vec{E} \times (\nabla \times \vec{E}) - c^2 \vec{B} \times (\nabla \times \vec{B}) \right] - \epsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B})$$

Then:

$$\boxed{\frac{d \vec{P}_{\text{mech}}}{dt}} = - \epsilon_0 \int_V d^3x \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) +$$

$$(6.114) \quad + \epsilon_0 \int_V d^3x \left[\vec{E} (\nabla \cdot \vec{E}) + c^2 \vec{B} (\nabla \cdot \vec{B}) \right. \\ \left. - \vec{E} \times (\nabla \times \vec{E}) - c^2 \vec{B} \times (\nabla \times \vec{B}) \right] \quad (6.116)$$

The term $-\epsilon_0 \int_V d^3x \frac{\partial}{\partial t} (\vec{E} \times \vec{B})$ can be passed to the left

and can be associated with the $\frac{\partial}{\partial t}$ of the electromagnetic momentum

\vec{P} field in the volume V. Then:

$$\vec{P}_{\text{field}} = \epsilon_0 \int_V d^3x (\vec{E} \times \vec{B}) = \underbrace{\mu_0 \epsilon_0}_{1/c^2} \int_V d^3x (\vec{E} \times \vec{H}) \quad (6.117)$$

Then: $\frac{1}{c^2} (\vec{E} \times \vec{H}) = \vec{P} = \begin{matrix} \text{density of electromag.} \\ \text{momentum} \end{matrix}$

$$(6.118)$$

So on the left we have:

$$\frac{\partial}{\partial t} (\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}).$$

What happens on the right of (6.116) ?

Consider $\vec{E}(\nabla \cdot \vec{E}) - \vec{E} \times (\nabla \times \vec{E}) =$

$$= \left[E_1 \left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3} \right) + E_2 \left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} \right) + E_3 \left(\frac{\partial E_3}{\partial x_3} - \left(\frac{\partial E_2}{\partial x_1} \right) \right) \right] \vec{e}_1$$

$$+ \dots \vec{e}_2 + \dots \vec{e}_3 = \text{see } A \text{ next page}$$

$$\nabla \times \vec{E} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ E_1 & E_2 & E_3 \end{vmatrix} = \vec{e}_1 \left(\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3} \right) + \vec{e}_2 \left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} \right) + \vec{e}_3 \left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} \right)$$

$$-\vec{E} \times (\nabla \times \vec{E}) = - \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ E_1 & E_2 & E_3 \\ \frac{\partial E_1}{\partial x_2} - \frac{\partial E_2}{\partial x_3} & \frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} & \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} \end{vmatrix} =$$

$$= - \left[E_2 \left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} \right) - E_3 \left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} \right) \right] \vec{e}_1 + \dots$$

$$\begin{aligned}
 A &= \underbrace{\frac{\partial E_1^2}{\partial x_1} + \frac{\partial}{\partial x_2} (E_1 E_2) + \frac{\partial}{\partial x_3} (E_1 E_3)}_{2 E_1 \frac{\partial E_1}{\partial x_1} + \frac{\partial E_1}{\partial x_2} \cdot E_2 + E_1 \frac{\partial E_2}{\partial x_2} + E_1 \frac{\partial E_3}{\partial x_3} + \frac{\partial E_1}{\partial x_3} \cdot E_3} - \frac{1}{2} \frac{\partial}{\partial x_1} (E_1^2 + E_2^2 + E_3^2) \\
 &\quad = \underbrace{- E_1 \frac{\partial E_1}{\partial x_1}}_{= E_1 \frac{\partial E_1}{\partial x_1}} - \underbrace{E_2 \frac{\partial E_2}{\partial x_1}}_{= E_1 \frac{\partial E_2}{\partial x_2} + E_2 \frac{\partial E_1}{\partial x_2}} - \underbrace{E_3 \frac{\partial E_3}{\partial x_1}}_{= E_1 \frac{\partial E_3}{\partial x_3} + E_3 \frac{\partial E_1}{\partial x_3}} \\
 &= E_1 \frac{\partial E_1}{\partial x_1} + E_1 \frac{\partial E_2}{\partial x_2} + E_1 \frac{\partial E_3}{\partial x_3} - E_2 \frac{\partial E_2}{\partial x_1} + E_2 \frac{\partial E_1}{\partial x_2} \\
 &\quad + E_3 \frac{\partial E_3}{\partial x_1} - E_3 \frac{\partial E_1}{\partial x_3} \quad \checkmark
 \end{aligned}$$

Then:

$$\left[\vec{E} (\nabla \cdot \vec{E}) - (\vec{E} \times (\nabla \times \vec{E})) \right] = \sum_{\alpha} \frac{\partial}{\partial x_\alpha} \left[(E_\alpha E_\beta) - \frac{1}{2} (\vec{E} \cdot \vec{E}) \delta_{\alpha\beta} \right] \quad (6.119)$$

of course we can repeat for $\vec{B} (\nabla \cdot \vec{B}) - \vec{B} \times (\vec{\nabla} \times \vec{B})$
and we obtain:

$$T_{\alpha\beta} \stackrel{\text{def.}}{=} \epsilon_0 \left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2} (\vec{E} \cdot \vec{E} + c^2 \vec{B} \cdot \vec{B}) \delta_{\alpha\beta} \right]$$

$$\left[\frac{\partial}{\partial t} (\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}) \right] = \sum_{\alpha} \int_V d^3x \frac{\partial}{\partial x_\alpha} T_{\alpha\beta} \quad (6.121)$$

divergence operator

$$= \oint_S \sum_{\alpha} T_{\alpha\beta} n_\beta da \quad (6.122)$$

$\sum_{\beta} T_{\alpha\beta} n_{\beta}$ is the α th component of the flow of momentum (per unit area) across the surface S.

It plays the role of a force (per unit area)

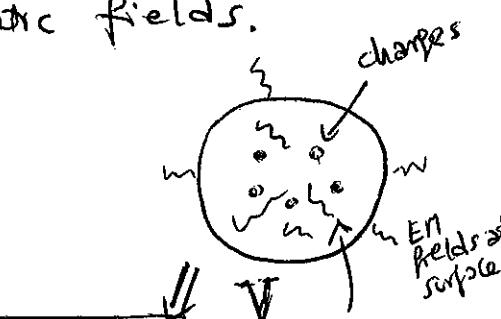
It can be used to calculate forces acting on material objects in electromagnetic fields.

In summary:

$$\frac{\partial}{\partial t} (\vec{P}_{\text{mech}} + \vec{P}_{\text{field}}) = \oint_S \sum_{\beta} T_{\alpha\beta} n_{\beta} d\sigma,$$

where $\vec{P}_{\text{field}} = \frac{1}{c^2} \int d^3x (\vec{E} \times \vec{H})$.

The framed equation represents the conservation of momentum.



"T is like a force per unit area acting on the surface"

$T_{\alpha\beta}$ is like a pressure.

(α, β)

Note: even perfectly static fields carry momentum as long as $\vec{E} \times \vec{H}$ is nonzero.