

7.1 Plane waves in a nonconducting medium

The Maxwell eqs. have traveling wave solutions that transport energy from one point to another. The simplest are transverse plane waves.

Consider the Max. eqs. without sources i.e. take Eqs. (6.6) and consider $\rho = 0$ and $\vec{J} = 0$.

$$\begin{aligned}\nabla \cdot \vec{B} &= 0, \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \\ \nabla \cdot \vec{D} &= 0, \quad \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = 0\end{aligned}\tag{7.1}$$

Assume solutions with "harmonic" t dependence $e^{-i\omega t}$

i.e. $\vec{E}(\vec{x}, t) = \vec{E}(\vec{x}) e^{-i\omega t}$, $\vec{D}(\vec{x}, t) = \vec{D}(\vec{x}) e^{-i\omega t}$

$$\frac{\partial \vec{D}}{\partial t} = -i\omega \vec{D}$$

$$\vec{B}(\vec{x}, t) = \vec{B}(\vec{x}) e^{-i\omega t}$$

$$\frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B}$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} - i\omega \vec{B} = 0$$

$$\nabla \cdot \vec{D} = 0, \quad \nabla \times \vec{H} + i\omega \vec{D} = 0$$

We assume a uniform isotropic linear media.

ϵ, μ are not \vec{x} dependent all three directions are the same

$\vec{D} = \epsilon \vec{E}$

$\vec{B} = \mu \vec{H}$

Then:

$$\nabla \times \vec{E} - i\omega \vec{B} = 0$$

$$\nabla \times \vec{B} + i\omega \mu \epsilon \vec{E} = 0$$

$$\underbrace{\nabla \cdot \nabla \times \vec{E}}_{=0} = \omega \nabla \cdot \vec{B}$$

Then $\nabla \cdot \vec{B} = 0$ is deduced from this group of two.

$$① \quad \nabla \times (\nabla \times \vec{E}) - i\omega \underbrace{\nabla \times \vec{B}}_{-i\omega \mu \epsilon \vec{E}} = 0$$

$$\underbrace{\nabla \times (\nabla \times \vec{E})}_{=0} + \mu \epsilon \omega^2 \vec{E} = 0$$

$$\nabla \underbrace{(\nabla \cdot \vec{E})}_{=0} - \nabla^2 \vec{E}$$

$$\underbrace{\nabla \cdot (\nabla \times \vec{B})}_{=0} = -i\omega \mu \epsilon \nabla \cdot \vec{E}$$

Then $\nabla \cdot \vec{D} = \nabla \cdot \vec{E} = 0$ is a consequence as well.

Then:
$$\boxed{(\nabla^2 + \mu \epsilon \omega^2) \vec{E} = 0} \quad (7.3)$$

$$② \quad \nabla \times (\nabla \times \vec{B}) + i\omega \mu \epsilon \underbrace{(\nabla \times \vec{E})}_{=0} = 0$$

$$\nabla \underbrace{(\nabla \cdot \vec{B})}_{=0} - \nabla^2 \vec{B} + i\omega \vec{B}$$

$$-\nabla^2 \vec{B} + \omega^2 \mu \epsilon \vec{B} =$$

$$\boxed{(\nabla^2 + \mu \epsilon \omega^2) \vec{B} = 0} \quad (7.3)$$

Consider 2 possible solution $e^{ikx-i\omega t}$.

$$\text{then, } \nabla^2 e^{ikx} = \nabla_x^2 e^{ikx} = (ik)^2 e^{ikx} = -k^2 e^{ikx}$$

$$\nabla^2(e^{ikx} e^{-i\omega t}) = -k^2(e^{ikx} e^{-i\omega t})$$

Then if $-k^2 + \mu\epsilon\omega^2 = 0$, we have 2 solution

$$k = \sqrt{\mu\epsilon} \omega \quad (7.4)$$

The "phase velocity" is defined as

$$v \stackrel{\text{def}}{=} \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n} \quad (7.5)$$

$$\frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\frac{\mu_0 \epsilon_0}{\mu_0 \epsilon_0}} = \sqrt{\frac{\mu_0 \epsilon_0}{\mu \epsilon}} \sqrt{c^2} = \underbrace{\sqrt{\frac{\mu_0 \epsilon_0}{\mu \epsilon}}}_{\text{def}} \cdot c = \frac{1}{n}$$

c = speed of light in vacuum

n = index of refraction

$\frac{c}{n}$ = speed of light in the medium

A more general solution is

$$\begin{aligned} u(x, t) &= a e^{ikx-i\omega t} + b e^{-ikx-i\omega t} \\ &= a e^{ik(x-vt)} + b e^{-ik(x+vt)} \end{aligned} \quad (7.6)$$

These are waves traveling in the positive and negative directions with velocity v .

We can superimpose them and create a general solution

From 2.44

$$f(x \pm vt) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(k) e^{ik(x \pm vt)} dk$$

\uparrow

$\rightarrow \infty$

Arbitrary function

Instead of focusing on "x", let us take an arbitrary wave vector $\vec{k} = k \hat{n}$.

$$\vec{E}(x, t) = \vec{E}_0 e^{ik\hat{n} \cdot \vec{x} - i\omega t}$$

$$\vec{B}(x, t) = \vec{B}_0 e^{ik\hat{n} \cdot \vec{x} - \omega t}$$

$$\nabla^2 \vec{E} = (ik\hat{n})^2 \vec{E} = -k^2 \hat{n} \cdot \hat{n} \vec{E}$$

$$\nabla^2 \vec{B} = -k^2 \hat{n} \cdot \hat{n} \vec{B}$$

Then we need $-k^2 \hat{n} \cdot \hat{n} + \mu \epsilon \omega^2 = 0$

or $k^2 \hat{n} \cdot \hat{n} = \mu \epsilon \omega^2$ (7.9)

But \hat{n} is a unit vector, thus $\hat{n} \cdot \hat{n} = 1$ and we go back to (7.4), $k = \sqrt{\mu \epsilon} \omega$.

However, we still need to make sure that

$$\nabla \cdot \vec{B} = 0 \text{ and } \nabla \cdot \vec{D} = 0 :$$

$$\nabla \cdot \vec{B} = ik\vec{n} \cdot \vec{B}_0 e^{ik\vec{n} \cdot \vec{x}} e^{-i\omega t} = 0$$

Then $\vec{n} \cdot \vec{B}_0 = 0$

(7.10)

Reciprocally from $\nabla \cdot \vec{D} = 0 \Rightarrow \vec{n} \cdot \vec{E}_0 = 0$

there are transverse waves since \vec{E} and \vec{B} are perpendicular to \vec{n} .

From: $\nabla \times \vec{E} = i\omega \vec{B}$

$$ik\vec{n} \times \vec{E} = i\omega \vec{B}$$

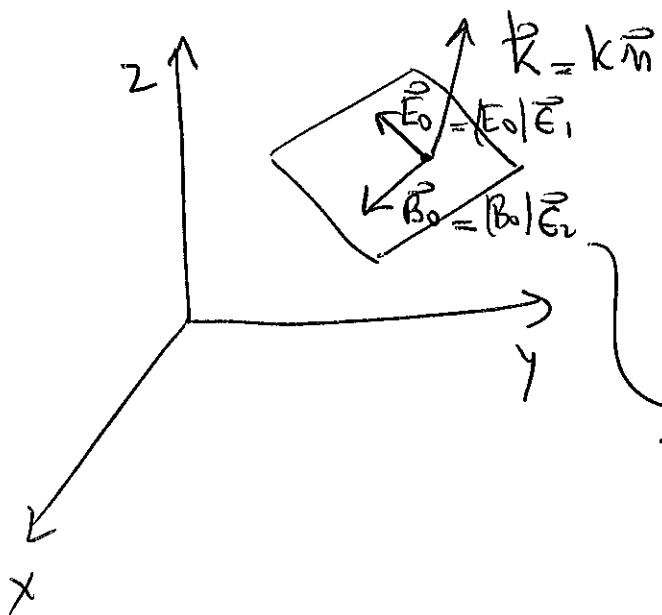
$$ik\vec{n} \times \vec{E}_0 = i\omega \vec{B}_0 \rightarrow \vec{B}_0 = \underbrace{\frac{k}{\omega}}_{(\mu\epsilon)} \vec{n} \times \vec{E}_0$$

Then: $C \vec{B}_0 = n (\vec{n} \times \vec{E}_0)$ (7.11)

$C \vec{B}_0$ and \vec{E}_0 have the same magnitude in free space ($n=1$)
and differ in n in a medium

same dimensions

Also note that \vec{n}, \vec{E}_0 , and \vec{B}_0 are orthogonal to each other



$\hat{n}, \hat{\epsilon}_1, \hat{\epsilon}_2$ are all unit vectors

$$= |E_0| \frac{m}{c} \hat{\epsilon}_2 \mp |E_0| \sqrt{\mu \epsilon} \hat{\epsilon}_2 \quad (7.12)$$

(7.5)

Let us consider how the Poynting vector

$$\vec{S} = \frac{1}{2} (\vec{E} \times \vec{H}^*) \quad \xleftarrow[\text{(see next page)}]{\text{time averaged}}$$

$\vec{E} = \vec{E}_0 e^{ik\vec{m} \cdot \vec{x}} e^{-int}$

$\vec{B} = \vec{B}_0 e^{ik\vec{m} \cdot \vec{x}} e^{-int}$

$$\vec{E} \times \vec{B}^* = \vec{E}_0 \times \vec{B}_0 = |\vec{E}_0| |\vec{B}_0| \sqrt{\mu\epsilon} |\vec{E}_0| \vec{E}_2 =$$

$$= \sqrt{\mu\epsilon} |\vec{E}_0|^2 \underbrace{(\vec{E}_1 \times \vec{E}_2)}_{\vec{n}} \quad (7.12)$$

$(|\vec{B}_0| = |\vec{E}_0| \frac{m}{c} = |\vec{E}_0| \sqrt{\mu\epsilon})$

$$\vec{S} = \frac{1}{2} (\vec{E} \times \vec{H}^*) = \frac{1}{2\mu} (\vec{E} \times \vec{B})^* = \frac{1}{2\mu} \sqrt{\mu\epsilon} |\vec{E}_0|^2 \vec{n}$$

$$= \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} |\vec{E}_0|^2 \vec{n} \quad (7.13)$$

The energy density is

$$\mu = \frac{1}{4} (\epsilon \vec{E} \cdot \vec{E}^* + \frac{1}{\mu} \vec{B} \cdot \vec{B}^*) \quad \xleftarrow[\text{(see next page)}]{\text{time averaged}}$$

$$= \frac{1}{4} \left(\epsilon |\vec{E}_0|^2 + \frac{1}{\mu} |\vec{B}_0|^2 \right) = \frac{\epsilon}{2} |\vec{E}_0|^2 \quad (7.14)$$

$\downarrow (7.12)$

$\mu \epsilon |\vec{E}_0|^2$

the ratio

$$\frac{|\vec{S}|}{\mu} = \frac{\frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} |\vec{E}_0|^2}{\frac{\epsilon}{2} |\vec{E}_0|^2} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\mu\epsilon}} = \frac{c}{n} = v$$

\uparrow

phase velocity

thus, the "phase velocity" is
the "speed of energy flow".

Note about time averages :

Suppose $f = a \cos(\vec{k} \cdot \vec{x} - ut)$
 $g = b \cos(\vec{k} \cdot \vec{x} - ut)$
 $(a, b \text{ real})$

Calculate $\langle fg \rangle =$

$$\overline{\langle fg \rangle} = \frac{1}{T} \int_0^T dt \quad fg =$$

$T = \frac{2\pi}{\omega}$

$$= \frac{1}{T} \int_0^T dt \quad ab \underbrace{\cos^2(\vec{k} \cdot \vec{x} - ut)}_{\cos^2 \varphi - \sin^2 \varphi = \cos^2 \varphi}$$

$$2\cos^2 \varphi - 1 = \cos^2 \varphi; \quad \cos^2 \varphi = \frac{1 + \cos 2\varphi}{2}$$

$$= \frac{1}{T} \int_0^T dt \quad ab \left[\frac{1}{2} + \frac{1}{2} \cos(2\vec{k} \cdot \vec{x} - 2ut) \right] =$$

$$= \frac{1}{T} T \frac{ab}{2} = \boxed{\frac{ab}{2}}$$

The average of a cosine is 0
over 2 period

In the complex notation

$$\tilde{f} = a e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

$$\tilde{g} = b e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

$$\tilde{f}\tilde{g}^* = ab ; \boxed{\frac{1}{2} \operatorname{Re}(\tilde{f}\tilde{g}^*) = \frac{ab}{2} = \langle fg \rangle}$$

So $\langle \vec{S} \rangle_{\text{time averaged}} = \operatorname{Re} \left\{ \frac{1}{2} (\vec{E} \times \vec{H}^*) \right\}$

$$\langle Eu \rangle = \frac{1}{2} \operatorname{Re} \left(\frac{\epsilon}{2} \vec{E} \cdot \vec{E}^* + \frac{1}{\mu} \vec{B} \cdot \vec{B}^* \right)$$

$$= \operatorname{Re} \left[\frac{1}{4} (\epsilon \vec{E} \cdot \vec{E}^* + \frac{1}{\mu} \vec{B} \cdot \vec{B}^*) \right]$$

7.2 Linear and Circular polarization

What we did before had the electric field always in the \vec{E}_1 direction. It is said to be linearly polarized with polarization vector \vec{E}_1 . Of course, we can also have a wave linearly polarized in the \vec{E}_2 direction.

$$\vec{E}_1 = \vec{E}_1 E_1 e^{i k \cdot \vec{x} - i \omega t}$$

$$\vec{E}_2 = \vec{E}_2 E_2 e^{i k \cdot \vec{x} - i \omega t}$$

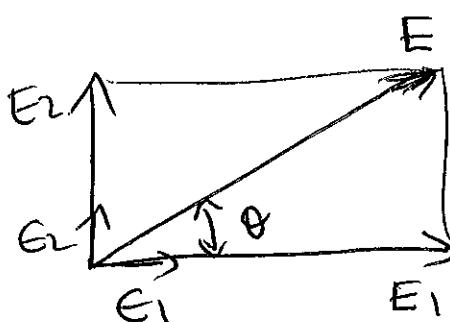
$$\text{From } C \vec{B}_0 = M \left(\vec{m} \times \vec{E}_0 \right) \\ = M \left(\vec{k} \times \vec{E}_0 \right)$$

$$\text{and } \vec{B}_{j=1,2} = \sqrt{\mu \epsilon} \frac{\vec{k} \times \vec{E}_j}{k} \quad (\text{from (7.11)})$$

If we linearly combine them and, moreover, we allow for a complex ~~phase~~ E_1 and E_2 so that we can have a phase difference between them, we get

$$\vec{E}(x, t) = (\vec{E}_1 \overline{E}_1 + \vec{E}_2 \overline{E}_2) e^{i k \cdot \vec{x} - i \omega t}$$

If E_1, E_2 have same phase (i.e. the same $e^{i\phi}$) then, the wave is linearly polarized with magnitude $\sqrt{E_1^2 + E_2^2}$ and angle $\theta = \tan^{-1} \left(\frac{E_2}{E_1} \right)$



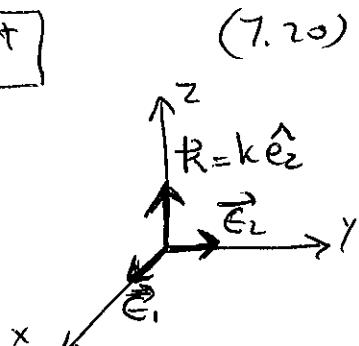
If E_1 and E_2 have different phases the wave is elliptically polarized. The simplest case is circular polarization when $E_1 = E_2$ and the phase differs in 90° .

Then: $\vec{E}(\vec{x}, t) = \underbrace{(\vec{E}_1 E_1 + \vec{E}_2 E_2)}_{\text{general}} e^{ik\vec{x} \cdot \vec{z} - \text{int}}$

circular $\cong (E_0 \vec{E}_1 + E_0 \vec{E}_2 e^{\pm i\pi/2}) e^{ik\vec{x} \cdot \vec{z} - \text{int}}$

$$= \overline{E_0 (\vec{E}_1 \pm i\vec{E}_2) e^{ik\vec{x} \cdot \vec{z} - \text{int}}} \quad (7.20)$$

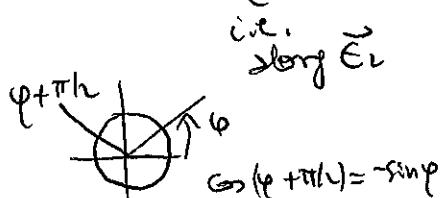
$$\text{Re } \vec{E}(\vec{x}, t) = E_0 \text{Re}((\vec{E}_1 \pm i\vec{E}_2) e^{ik\vec{x} \cdot \vec{z} - \text{int}})$$



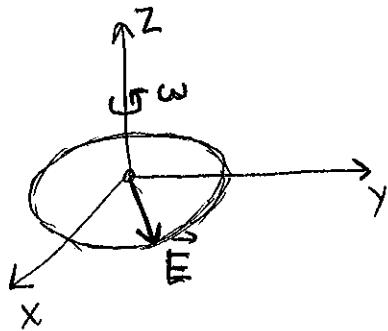
Along X: i.e. along \vec{E}_1

$$\text{Re } E_x(\vec{x}, t) = E_0 \cos(kz - \omega t) \quad (7.21)$$

$$\text{Re } E_y(\vec{x}, t) = E_0 \sin(kz - \omega t)$$



In this case, it is as if the electric field is rotating with angular velocity ω (and constant magnitude)



$E_1 + iE_2$

counterclockwise
(as in sketch)

$E_1 - iE_2$

clockwise

positive helicity
left circularly polarized
right c.p.
negative helicity

Defining $\vec{E}_{\pm} = \frac{1}{\sqrt{2}}(\vec{E}_1 \pm i\vec{E}_2)$

then a general wave can be written as

$$\vec{E}(x, t) = (E_+ \vec{e}_+ + E_- \vec{e}_-) e^{ik_x x - i\omega t} \quad (7.24)$$

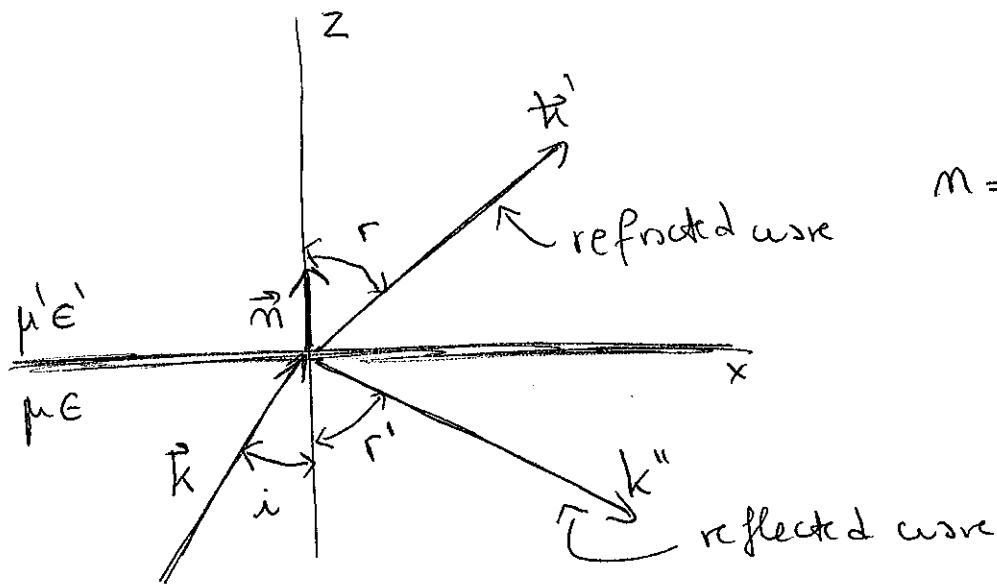
Complex

For circular polarization $\frac{E_-}{E_+} = \pm 1$

For elliptical polarization $\frac{E_-}{E_+} = r \neq \pm 1$

→ Skip Stokes parameters.

7.3 Reflection and Refraction of Waves



$$m = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}, \quad m' = \sqrt{\frac{\mu'\epsilon'}{\mu_0\epsilon_0}}$$

From (7.18)

Incident

$$\vec{E} = E_0 e^{i\vec{k} \cdot \vec{x} - i\omega t}$$

$$\vec{B} = \sqrt{\mu\epsilon} \left(\frac{\vec{k} \times \vec{E}}{k} \right)$$

Reflected

$$\vec{E}' = E'_0 e^{i\vec{k}' \cdot \vec{x} - i\omega t}$$

$$\vec{B}' = \sqrt{\mu'\epsilon'} \left(\frac{\vec{k}' \times \vec{E}'}{k'} \right)$$

Reflected

$$\vec{E}'' = E''_0 e^{i\vec{k}'' \cdot \vec{x} - i\omega t}$$

$$\vec{B}'' = \sqrt{\mu\epsilon} \left(\frac{\vec{k}'' \times \vec{E}''}{k''} \right)$$

same magnitude
as \vec{k} , but a
different direction.

but $k'' = k = \omega\sqrt{\mu\epsilon}$
while $k' = \omega\sqrt{\mu'\epsilon'}$

At $z=0$ the spatial and time variation of all fields must be the same. So this is a statement about the phases such as e^{ikz} . We must have:

$$\underbrace{(\mathbf{R} \cdot \hat{\mathbf{x}})}_{k_z z + k_{\parallel} x \mid z=0} = \underbrace{(\mathbf{R}' \cdot \hat{\mathbf{x}})}_{k' \sin(r) x} = \underbrace{(\mathbf{R}'' \cdot \hat{\mathbf{x}})}_{k'' \sin(r') x}$$

$$= k_{\parallel} x$$

$$= k \sin(i) x$$

From $k \sin(i) = k \sin(r') \Rightarrow i = r'$

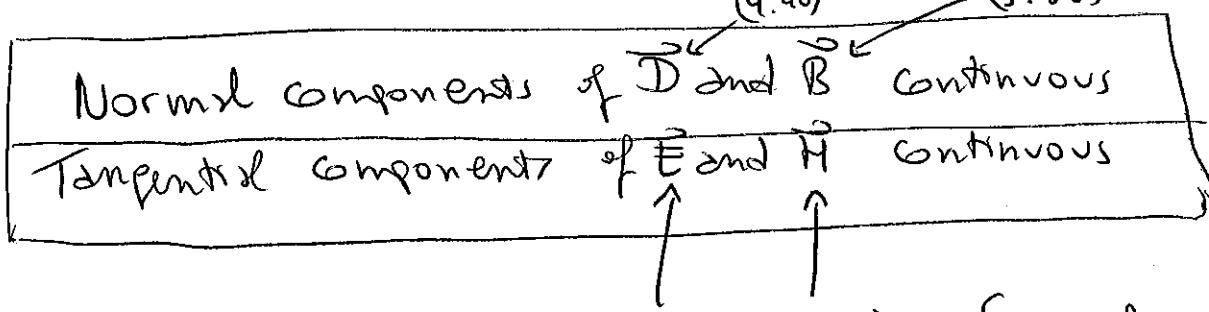
From $k \sin(i) = k' \sin(r) \Rightarrow$

$$\frac{\sin(i)}{\sin(r)} = \frac{k}{k'} = \frac{\omega_1 n_1}{\omega_2 n_2} = \frac{n'}{n}$$

\leftarrow Snell's laws

which are the well known laws from elementary physics. There are the so-called "kinematic properties".

But then we have "dynamic properties" that arise from the amplitudes in front of the phase factors.



See the
Section I.5
Page 16

The boundary conditions are:

$$[\epsilon(\vec{E}_0 + \vec{E}_0'') - \epsilon' \vec{E}_0'] \cdot \vec{n} = 0 \quad \text{normal of } \vec{D} \text{ cont.}$$

$$(\hat{k} \times \vec{E}_0 + \hat{k}'' \times \vec{E}_0 - \hat{k}' \times \vec{E}_0') \cdot \vec{n} = 0 \quad \text{normal of } \vec{B} \text{ cont.}$$

↑ ↑ ↑
common factors: $\frac{\mu\epsilon}{k} = \frac{1}{\omega}$ $\frac{\mu'\epsilon'}{k'} = \frac{1}{\omega}$

$$\left(\frac{\mu\epsilon}{k} = \frac{1}{\omega}, \frac{\mu'\epsilon'}{k'} = \frac{1}{\omega}, \frac{\mu'\epsilon'}{k'} = \frac{1}{\omega} \right) \quad (7.37)$$

$$(\vec{E}_0 + \vec{E}_0'' - \vec{E}_0') \times \vec{n} = 0 \quad \text{tangential of } \vec{E} \text{ cont.}$$

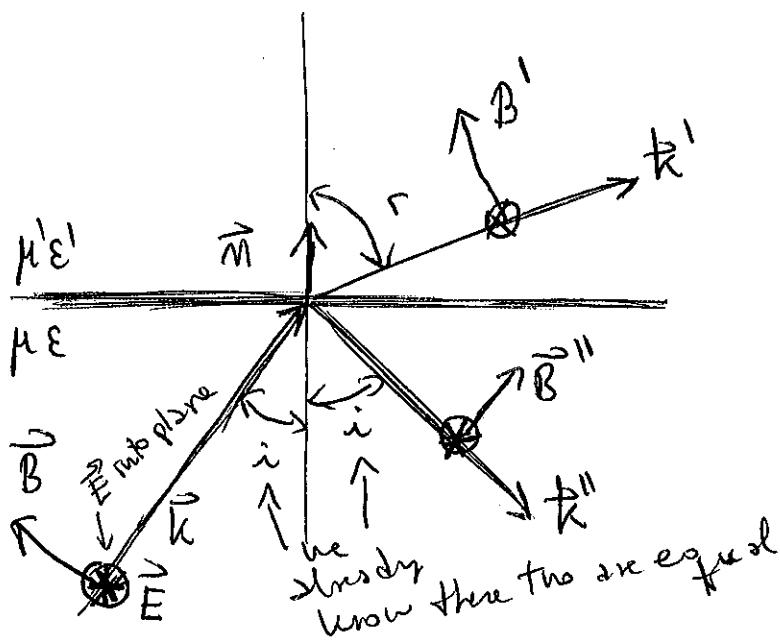
$$\left(\frac{\hat{k} \times \vec{E}_0 + \hat{k}'' \times \vec{E}_0'' - \hat{k}' \times \vec{E}_0'}{\mu} \right) \times \vec{n} = 0 \quad \text{tangential of } \vec{H} \text{ cont.}$$

$$\vec{H} = \frac{\vec{B}}{\mu}$$

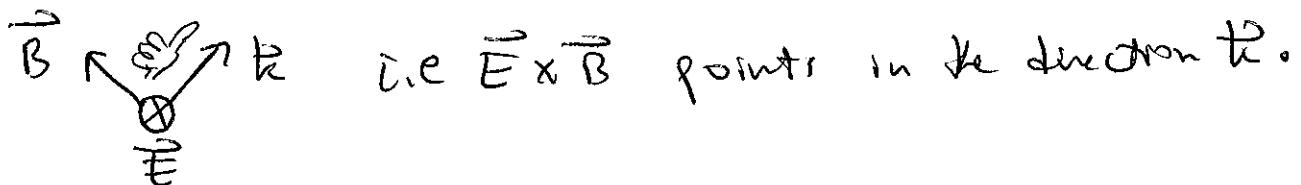
Similar to normal of \vec{B} cont.
but using \vec{H} instead

Let us consider two special cases:

(1) Electric field \perp to the plane of incidence



The "plane of incidence" is defined by \vec{k} and \vec{n} . Thus, " \perp to the plane of incidence" means that the \vec{E} field points either in or out of the previous figure. We choose \vec{B} such that $\vec{E}, \vec{B}, \vec{k}$ are mutual orthogonal vectors.



In this case, $\vec{E}_0, \vec{E}_0', \vec{E}_0''$ are all \perp to \vec{n} and the first boundary condition gives nothing.

The third gives:

$$\left\{ \begin{array}{l} \vec{E}_0 \times \vec{n} \\ \vec{E}_0' \times \vec{n} \\ \vec{E}_0'' \times \vec{n} \end{array} \right\} \text{ all point in the same direction}$$

$\boxed{\vec{E}_0 + \vec{E}_0'' - \vec{E}_0' = 0}$

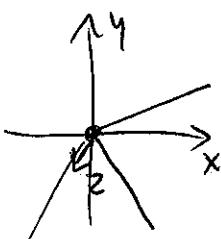
no need to use vector notation since
all point in the same direction

(7.38)

The fourth BC:

$$(\vec{k} \times \vec{E}_0) \times \vec{n} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ k_x & k_y & 0 \\ 0 & 0 & -E_0 \end{vmatrix} \times \vec{n} = \begin{pmatrix} -k_y E_0 \\ +k_x E_0 \\ 0 \end{pmatrix} \times \vec{n} =$$

$$= \begin{pmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ -k_y E_0 & k_x E_0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = -k \cos(i) E_0 \hat{e}_z$$



$$(\vec{k}' \times \vec{E}_0') \times \vec{n} = -k \cos(r) E_0' \hat{e}_z$$

↑ transmitted

$$\begin{aligned}
 (\hat{k}'' \times \vec{E}_0'') \times \hat{n} &= \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ k_x'' & k_y'' & 0 \\ 0 & 0 & -E_0'' \end{vmatrix} \\
 &\quad \text{reflected} \\
 k_x'' &= k'' \sin(i) \\
 k_y'' &= k'' \cos(i)
 \end{aligned}$$

$$\begin{aligned}
 \cancel{\begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ -k_y E_0 & k_x E_0 & 0 \\ 0 & 0 & 0 \end{vmatrix}} \\
 &= -k_y'' E_0'' \hat{e}_z \\
 &= k'' \cos(i) E_0'' \hat{e}_z
 \end{aligned}$$

Then:

$$\frac{-k \cos(i) E_0}{\mu} + \cancel{\frac{k'' \cos(i) E_0''}{\mu}} - \frac{(-k' \cos(r) E_0')}{\mu'} = 0$$

$$(E_0 - E_0'') \cos(i) \left(\cancel{\frac{-k}{\mu}} + \cancel{\frac{k'}{\mu'}} \right) + \cancel{\left(\frac{k'}{\mu'} \right)} \cos(r) E_0' = 0$$

But $k = \omega \sqrt{\mu \epsilon} = k''$ // $k' = \omega \sqrt{\mu' \epsilon'}$

$$\frac{\omega \sqrt{\mu \epsilon}}{\mu} = \omega \sqrt{\frac{\epsilon}{\mu}}$$

Then:

$$\boxed{\frac{\epsilon}{\mu} (E_0 - E_0'') \cos(i) - \frac{\epsilon'}{\mu'} \cos(r) E_0' = 0} \quad (7.38)$$

The second BC duplicates the third (after using (7.36))
 (left as exerchin).

From (7.38) $E_0 + E_0'' - E_0' = 0 \Rightarrow E_0'' = E_0' - E_0$

Back to the second (7.38):

$$\sqrt{\frac{\epsilon}{\mu}} \underbrace{(E_0 - (E_0' - E_0))}_{2E_0 - E_0'} \cos(i) = \sqrt{\frac{\epsilon'}{\mu'}} \cos(r) E_0'$$

$$2\sqrt{\frac{\epsilon}{\mu}} \cos(i) E_0 = \left(\sqrt{\frac{\epsilon'}{\mu'}} \cos(r) + \sqrt{\frac{\epsilon}{\mu}} \cos(i) \right) E_0'$$

$$\frac{E_0'}{E_0} = \frac{2\sqrt{\frac{\epsilon}{\mu}} \cos(i)}{\sqrt{\frac{\epsilon}{\mu}} \cos(i) + \sqrt{\frac{\epsilon'}{\mu'}} \cos(r)} =$$

$$= \frac{2 \cos(i)}{\cos(i) + \left(\frac{\epsilon'}{\epsilon} \frac{m}{m'} \right) \frac{1}{\sqrt{m'^2 - m^2 \sin^2 i}}}.$$

$\frac{m'}{m} \mu (7.36)$

$$\begin{aligned} \sqrt{1 - \sin^2 i} &= \sqrt{1 - \left(\sin(i) \frac{m}{m'} \right)^2} = \\ (7.36) \quad &= \sqrt{m'^2 - m^2 \sin^2 i} \frac{1}{m'} \end{aligned}$$

$$\boxed{\frac{2 m \cos(i)}{m \cos(i) + \frac{\mu}{\mu'} \sqrt{m'^2 - m^2 \sin^2(i)}} = \frac{E_0'}{E_0}} \quad (\text{top 7.39})$$

(bottom 7.39) ↗

$$\text{From } E_0'' = E_0' - E_0 \Rightarrow \boxed{\frac{E_0''}{E_0} = \frac{E_0'}{E_0} - 1 = \frac{m \cos(i) - \frac{\mu}{\mu'} \sqrt{m'^2 - m^2 \sin^2(i)}}{m \cos(i) + \frac{\mu}{\mu'} \sqrt{m'^2 - m^2 \sin^2(i)}}}$$