In previous sections we did not consider "dispersion" and for this reason an arbitrary wave train will travel without distortion. However, in reality there is dispersion.

If the calculations involve a single ω, nothing changes. Simply μ and ε are those of the ω under study.
But suppose we superimpose a range of frequencies: then we have to use here a $\varepsilon = \varepsilon(\omega)$, $\mu = \mu(\omega)$.
New effects may arise once this ω dependence is added.

A. Simple model

As in Sec. 4.6 represent an electron as bounded to the positive charge on an atom by a "spring" of frequency ω_0

$$\vec{F} = -m\omega_0^2 \vec{x} \qquad (4.71)$$

Now introduce an electric field that adds a force $e\vec{E}$

In equilibrium $|\vec{F}| = m\omega_0^2 |\vec{x}| = e|\vec{E}|$

Induced moment:

$$\vec{p} = e\vec{x} = e \cdot \frac{e}{m\omega_0^2}\vec{E}$$

polarizability "γ"

So the analysis of Sect.(4.71) basically focused on a constant $\vec{E}$ and a new equil. position. But what happens if $\vec{E}$ is ω dependent?

$$m\ddot{\vec{x}} = -m\omega_0^2\vec{x} - e\vec{E}(\vec{x},t)$$

↑ we assume the field is ≈ constant in space.

Assume $\vec{E} = \vec{E}_0 e^{-i\omega t}$.

Assume $\vec{x}$ has the same frequency: $\vec{x} = \vec{x}_0 e^{-i\omega t}$

$$\ddot{\vec{x}} = \vec{x}_0(-\omega^2)e^{-i\omega t} = -\omega^2\vec{x}$$

Then:

$$-m\omega^2\underbrace{\vec{x}_0 e^{-i\omega t}}_{\vec{x}(\omega)} = -m\omega_0^2\underbrace{\vec{x}_0 e^{-i\omega t}}_{\vec{x}(\omega)} - e\underbrace{\vec{E}_0 e^{-i\omega t}}_{\vec{E}(\omega)}$$

$$m(\omega_0^2-\omega^2)\vec{x}(\omega) = -e\vec{E}(\omega)$$

and

$$\boxed{\vec{p}(\omega) = -e\vec{x}(\omega) = \frac{e^2\vec{E}(\omega)}{m(\omega_0^2-\omega^2)}} \qquad (7.50)$$

induced moment $\vec{p}=\vec{p}(\omega)$

This is the contribution of just <u>one</u> electron.

if we add "damping"

$\omega_0^2-\omega^2 \Rightarrow \omega_0^2-\omega^2-i\omega\gamma$

(left as exercise)

Let us consider N molecules. Then:

$$\vec{P}(\omega) = \frac{Ne^2}{m(\omega_0^2-\omega^2)}\vec{E}(\omega)$$

But we know that $\vec{P} = \epsilon_0 \chi_e \vec{E}$ (4.36)

Then: $$\chi_e = \frac{Ne^2}{\epsilon_0 m(\omega_0^2-\omega^2)}$$

Since (4.38) says $\epsilon = \epsilon_0(1+\chi_e)$ then:

$$\boxed{\frac{\epsilon(\omega)}{\epsilon_0} = 1+\chi_e = 1 + \frac{Ne^2}{\epsilon_0 m(\omega_0^2-\omega^2)}}$$

(7.51) if $\gamma=0$ $f_j=1$, $j=1$

which is (7.51) if $\gamma=0$
and if it is assumed just one e^-
per molecule contributes
i.e. j is only 1 and $f_{j=1}=1$.

If we try to have a more "QM" description then we add many frequencies in a phenomenological way:

integers representing # of e^- at each level

(7.51) if $\gamma=0$

$$\boxed{\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m}\sum_j f_j \frac{1}{\omega_j^2-\omega^2}}$$

$Z = \sum_j f_j$ is the total # of electrons in the molecule.

If we consider an insulator, each electron is bounded to an atom or molecule namely $\omega_0 \neq 0$. Then, when $\omega \to 0$, we recover results that intuitively are fine, namely $\epsilon > 1$, real, and the imaginary parts are not important. I.e. same as in Chapter 4.

But what happens if some electrons are "free" or almost free like conduction electrons? then, it is as if ω_0 is ≈ 0 for them.

In such a case:

$$\frac{\epsilon(\omega)}{\epsilon_0} = \overbrace{1 + \underbrace{\frac{Ne^2}{\epsilon_0 m}\sum_j f_j \frac{1}{\omega_j^2-\omega^2}}_{\text{bounded } e^-\text{, thus I could take this in the limit } \omega=0}}^{\frac{\epsilon_{rest}(\omega)}{\epsilon_0}} + \underbrace{\frac{Ne^2}{\epsilon_0 m}\frac{f_0}{(\omega_0^2-\omega^2-i\omega\gamma_0)}}_{\text{Take } \omega_0=0 \text{ here:}}$$

$$\frac{Ne^2 f_0}{\epsilon_0 m\,(-\omega^2 - i\omega\gamma_0)}$$

$$i\omega \cdot i\omega - i\omega\gamma_0 = i\omega\, i\left(\omega - \frac{\gamma_0}{i}\right) = i\omega(i\omega-\gamma_0) = -i\omega(\gamma_0 - i\omega) = \frac{\omega}{i}(\gamma_0 - i\omega)$$

$$\frac{Ne^2 f_0}{\epsilon_0 m}\frac{i}{\omega(\gamma_0 - i\omega)}$$

$$\boxed{\epsilon(\omega) = \epsilon_{rest}(\omega) + i\,\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}} \qquad (7.56)$$

Let us introduce $\sigma \equiv \frac{Ne^2 \tau_0}{m(\gamma_0 - i\omega)}$

Then: $\epsilon(\omega) = \epsilon_{rest}(\omega) + \frac{i\sigma(\omega)}{\omega}$

Consider one of Maxwell's equations

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

conductivity. We pretend we do not know the relation with σ above.

and assume that $\boxed{\vec{J} = \sigma \vec{E}}$ i.e. that it obeys the Ohm's law, and that it also has a normal dielectric constant ϵ_{rest} (or ϵ_b as in the book)

Then:

assume ω-indep.

$$\nabla \times \vec{H} = \sigma \vec{E} + \frac{\partial(\epsilon_b \vec{E})}{\partial t}$$

$$= \sigma \vec{E} + \epsilon_b \frac{\partial \vec{E}}{\partial t}$$

assume $\vec{E} = \vec{E}_0 e^{-i\omega t}$, $\frac{\partial \vec{E}}{\partial t} = -i\omega \vec{E}$

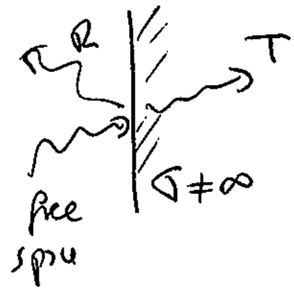
$$= \left(\sigma + \epsilon_b(-i\omega)\right)\vec{E} = \boxed{-i\omega\left(\epsilon_b + \frac{i\sigma}{\omega}\right)\vec{E}} \qquad (7.57)$$

Then σ is like a conductivity, as the notation implied.

So at $\omega \approx 0$, the difference metal vs. insulator are dramatic. But at large ω, they are very similar.

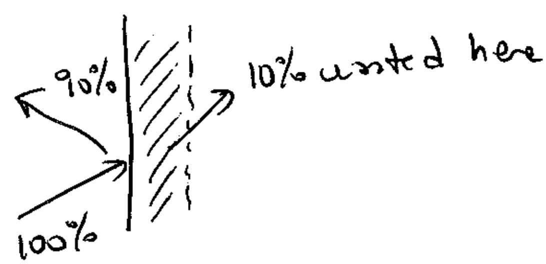
Note that for the case $\sigma \neq \infty$, there must be some "resistance" and some power must be dissipated in heat.

Then, in general $R + T \neq 1$.



Problem 7.5 shows that in the case sketched, actually $T = 0$ because the wave a decay in its amplitude as it penetrates the conductor.

But $R < 1$, because the penetrating wave dissipates some energy. Example:



If $\sigma \to \infty$ then, $R = 1$, $T = 0$. No energy wasted. Electric field does not penetrate in a perfect metal.

For an excellent discussion see Secs. 8.3.2 and 8.3.3 of Griffiths. This actually should be a Homework problem.