Chapter 9

We turn now to the emission of radiation by localized systems of oscillating charge and current densities. Namely, we focus on the origin of the plane waves that were discussed in previous chapters.

9.1 Localized oscillating sources

We will focus on just one frequency ω, since other time dependences can be handled by Fourier analysis.

Thus, we assume:

We will not worry much about "dispersion" since most of the work will be in "vacuum".

$$\rho(\vec{x},t) = \rho(\vec{x})\, e^{-i\omega t}$$
$$\vec{J}(\vec{x},t) = \vec{J}(\vec{x})\, e^{-i\omega t} \tag{9.1}$$

Electromagnetic potentials and fields are also assumed to have the same t-dependence. For simplicity, the sources are assumed to be located in empty space.

From Ch.6, we know that:

$$A(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \int dt' \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\, \delta\left(t' + \frac{|\vec{x}-\vec{x}'|}{c} - t\right) \tag{9.2}$$

the δ-function enforces the causal behavior of the fields.

Assuming $\vec{J}(\vec{x}',t') = \vec{J}(\vec{x}')\, e^{-i\omega t'}$, then

$$\underbrace{\vec{A}(\vec{x},t)}_{\vec{A}(\vec{x})e^{-i\omega t}} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} \, e^{-i\omega t} \, e^{+i\left(\frac{\omega}{c}\right)|\vec{x}-\vec{x}'|} .$$

$\frac{\omega}{c} \rightarrow k$ wave number

Thus:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')\, e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} , \qquad (9.3)$$

and

$$\vec{H} = \frac{\vec{B}}{\mu_0} = \frac{1}{\mu_0}\left(\nabla \times \vec{A}\right) . \qquad (9.4)$$

<u>Outside the source</u> i.e. where $\vec{J}(\vec{x}) = 0$,

we use one of Maxwell's eqs.

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (6.6)$$

with $\vec{D} = \epsilon_0 \vec{E}$. Outside source, and assuming $\vec{E}$ varies as $e^{-i\omega t}$, then:

$$\underbrace{\nabla \times \vec{H}}_{\vec{H}(\vec{x})e^{-i\omega t}} = \epsilon_0 (-i\omega)\, \vec{E}(\vec{x})\, e^{-i\omega t}$$

Then:

$$\vec{E}(\vec{x}) = \frac{i}{\epsilon_0 \omega} \nabla \times \vec{H}(\vec{x}) = \frac{i}{\epsilon_0 k c}\left(\nabla \times \vec{H}(\vec{x})\right)$$

But $\epsilon_0\mu_0 = \frac{1}{c^2}$, then:

$$\vec{E}(\vec{x}) = \frac{i}{\epsilon_0 k}\sqrt{\epsilon_0\mu_0}\left(\nabla\times\vec{H}(\vec{x})\right),$$

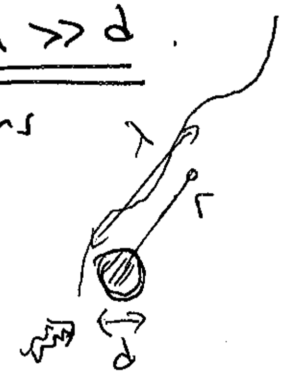
$$= \frac{i}{k} Z_0\left(\nabla\times\vec{H}(\vec{x})\right). \qquad (9.5)$$

$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$ called "impedance of free space"

Then, in principle we are set: given $\vec{J}(\vec{x})$ then we can get $\vec{A}$, then $\vec{H}$, and then $\vec{E}$. But the math can be complicated. Thus, let us consider a few limits to simplify the situation:

Suppose the sources are confined to a small region of size "d" and suppose the wavelength is $\lambda = \frac{2\pi c}{\omega}$ and $\underline{\lambda \gg d}$.

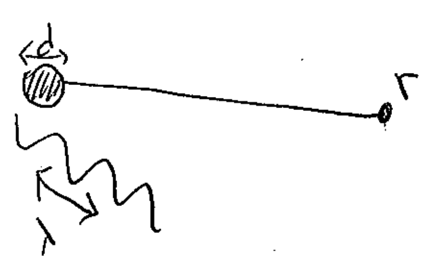
In this case there are 3 regions of relevance:



$d \ll r \ll \lambda$ ← near

$d \ll r \sim \lambda$ ← intermediate

$d \ll \lambda \ll r$ ← far



In the "near" zone, since $d \ll r \ll \lambda$ then k is "small" since $k = \frac{2\pi}{\lambda}$ and λ is the largest number. Then

$$\boxed{e^{ik|\vec{x}-\vec{x}'|} \approx 1}$$ because $|\vec{x}-\vec{x}'|$ is of order r and $kr = \frac{2\pi}{\lambda} r \ll 1$.

In this case

$$\vec{A}(\vec{x}) \underset{(9.3)}{=} \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$ which is (5.32) that was deduced in the <u>magnetostatic</u> context.

Then in this limit the fields are <u>quasistationary</u> <u>they are those of the static problem just oscillating in time via $e^{-i\omega t}$</u>. So all we learned before applies here. (λ <u>large</u> is like ω <u>small</u>, thus the time dependence is kind of weak).

What happens in the other extreme: "far" i.e. $d \ll \lambda \ll r$. Since $\lambda = \frac{2\pi}{k}$ this means

$$\boxed{kr \gg 1}. \quad \left[\text{or } kr = \frac{2\pi}{\lambda} r = 2\pi\left(\frac{r}{\lambda}\right)\right]$$

the exponential $e^{ik\underbrace{|\vec{x}-\vec{x}'|}}$ oscillates rapidly.

order r since $|\vec{x}'|$ is small (localized charge)

$$|\vec{x}-\vec{x}'| = |\vec{x}|\left|\underbrace{\frac{\vec{x}}{|\vec{x}|}}_{\vec{n}} - \frac{\vec{x}'}{|\vec{x}|}\right| = |\vec{x}|\sqrt{\left(\vec{n}-\frac{\vec{x}'}{|\vec{x}|}\right)\cdot\left(\vec{n}-\frac{\vec{x}'}{|\vec{x}|}\right)}$$

$$\cong |\vec{x}|\left(1 - 2\frac{\vec{n}\cdot\vec{x}'}{|\vec{x}|}\right)^{1/2} = |\vec{x}|\left(1 - \frac{\vec{n}\cdot\vec{x}'}{|\vec{x}|}\right) =$$

$$= \overbrace{|\vec{x}|}^{r} - \vec{n}\cdot\vec{x}' = r - \vec{n}\cdot\vec{x}' \quad (9.7)$$

$$e^{ik|\vec{x}-\vec{x}'|} \approx e^{ikr}\, e^{-ik\vec{n}\cdot\vec{x}'} \quad \text{and} \quad \frac{1}{|\vec{x}-\vec{x}'|} \approx \frac{1}{r};$$

$$\boxed{\vec{A}(\vec{x}) \underset{kr\to\infty}{\cong} \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')\, e^{-ik\vec{n}\cdot\vec{x}'}\, d^3x'} \qquad (9.8)$$

crucial $\frac{1}{r}$ for radiation problems

Using $e^{-ik\vec{n}\cdot\vec{x}'} = \sum_{n}^{\infty} \frac{1}{n!}(-ik)^n (\vec{n}\cdot\vec{x}')^n$,

we can simplify further by keeping only the first terms

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_{n}\frac{(-ik)^n}{n!}\int \vec{J}(\vec{x}')(\vec{n}\cdot\vec{x}')^n\, d^3x'} \qquad (9.9)$$

$|\vec{x}'| \sim d$ (source size)

$k^n d^n$ decays fast with n because $kd \ll 1$ since we assumed $d \ll \lambda = \frac{2\pi}{k}$.

Keeping just one or two terms is often sufficient.