9.2 Electric Dipole Fields and Radiation

Let us keep just the first term in the expansion i.e. $n=0$. Then we get:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')\, d^3x'$$

Consider the integral $\vec{I} = \int_V \vec{x}'(\nabla'\cdot\vec{J}(\vec{x}'))\, d^3x'$

Use the component "1" as example

$$I_1 = \int x'_1(\nabla'\cdot J)\, d^3x' = \int x'_1\left(\frac{\partial J_1}{\partial x'_1} + \frac{\partial J_2}{\partial x'_2} + \frac{\partial J_3}{\partial x'_3}\right) d^3x'$$

Note that $\int x'_1 \frac{\partial J_2}{\partial x'_2}\, d^3x' = \int \frac{\partial}{\partial x'_2}(x'_1 J_2)\, d^3x'$ since $\frac{\partial x'_1}{\partial x'_2} = 0$

and the same for $\int x'_1 \frac{\partial J_3}{\partial x'_3}\, d^3x'$.

The case $\int x'_1 \frac{\partial J_1}{\partial x'_1}\, d^3x'$ leads to $\int \frac{\partial}{\partial x'_1}(x'_1 J_1)\, d^3x' =$

$$= \int J_1\, d^3x' + \int x'_1 \frac{\partial J_1}{\partial x'_1}\, d^3x'$$

Then:

$$I_1 = -\int J_1\, d^3x' + \underbrace{\int \left[\frac{\partial}{\partial x'_1}(x'_1 J_1) + \frac{\partial}{\partial x'_2}(x'_1 J_2) + \frac{\partial}{\partial x'_3}(x'_1 J_3)\right] d^3x'}_{\int_V \nabla\cdot(x'_1\vec{J})\, d^3x' = \oint_S x'_1 \vec{J}\cdot\vec{n}'\, da'}$$

$= 0$ for localized sources.

Then $\quad I_1 = -\int J_1(\vec{x}')\, d^3x'$

and the same holds for components 2 and 3. Thus:

$$\left[\int_V \vec{J}(\vec{x}')\, d^3x' = -\int_V \vec{x}'\,[\nabla'\cdot\vec{J}(\vec{x}')]\, d^3x'\right]$$

Moreover, from $\quad \nabla\cdot\vec{J} + \underbrace{\frac{\partial \rho}{\partial t}}_{} = 0 \quad (6.3)$

$\vec{J}(\vec{x})\, e^{-i\omega t}$

$\frac{\partial}{\partial t}[\rho(\vec{x})\, e^{-i\omega t}] =$

$= -i\omega\, \rho(\vec{x})\, e^{-i\omega t}$

we get:

$$\boxed{\int \vec{J}(\vec{x}')\, d^3x' = -i\omega \int \vec{x}'\, \rho(\vec{x}')\, d^3x'} \qquad (9.14)$$

Thus:

$$\vec{A}(\vec{x}) \overset{n=0}{\approx} -\frac{i\mu_0\omega}{4\pi}\, \frac{e^{ikr}}{r} \underbrace{\int \vec{x}'\, \rho(\vec{x}')\, d^3x'}$$

but this the electric dipole moment $\vec{p}$ defined before in electrostatics

Note: for localized charges, the total charge cannot change with time since it is conserved. Thus, it does not contribute. See Jackson 410 (page) for a discussion in more detail.

Then:

$$\boxed{\vec{A}(\vec{x}) = -\frac{i\mu_0\omega}{4\pi}\, \vec{p}\, \frac{e^{ikr}}{r}}$$

Note that for $\omega = 0$ then it cancels.
So only for time-dependent phenomena this matters. → Even if $\frac{\partial \rho}{\partial t} = 0$, we may still have a current that satisfies $\nabla \cdot \vec{J} = 0$, but it was shown in Chapter 5 that $\int \vec{J}\, d^3x' = 0$ in such a case.

The magnetic field is given by (9.4):

$$\vec{H} = \frac{1}{\mu_0}(\nabla \times \vec{A}) \overset{n=0}{\approx} -\frac{i\omega}{4\pi}\, \nabla \times \left(\vec{p}\, \frac{e^{ikr}}{r}\right)$$

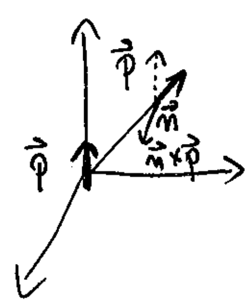
Let us use: $\nabla \times (\psi \vec{C}) = (\nabla\psi) \times \vec{C} + \psi(\nabla \times \vec{C})$

If $\vec{C} = \vec{p}$ and since $\vec{p}$ is just a vector number then $\nabla \times \vec{C} = 0$. We only have "$\nabla\psi$" $\neq 0$.

$$\nabla\left(\frac{e^{ikr}}{r}\right) \underset{\text{backcover}}{=} \hat{e}_r \frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right) = \underset{\vec{n}}{\hat{e}_r}\left(ik\, \frac{e^{ikr}}{r} - \frac{1}{r^2} e^{ikr}\right)$$

$$= \vec{n}\, \frac{e^{ikr}}{r}\, ik \left(1 - \frac{1}{ikr}\right)$$

$$\vec{H} = \underbrace{\frac{-i\omega}{4\pi}\frac{ik}{r}}_{\omega k \,\underset{\frac{\omega}{c}=k}{=}\, ck^2} e^{ikr}\left(1 - \frac{1}{ikr}\right)(\vec{n} \times \vec{p})$$



$$\boxed{\vec{H} = \frac{ck^2}{4\pi}(\vec{n} \times \vec{p})\, \frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)} \qquad (9.18)$$

Proof of (9.18) (electric field)

$$\boxed{\vec{E} = \frac{i Z_0}{k}\left(\nabla \times \vec{H}\right) = \frac{i Z_0}{k}\, \nabla \times \left[\frac{ck^2}{4\pi}(\vec{n}\times\vec{p})\underbrace{\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)}_{f(r)}\right]} \qquad Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$= \frac{i Z_0 . ck}{4\pi}\; \nabla \times \left[(\vec{n}\times\vec{p})\, f(r)\right]$$

Let us use now $\nabla\times(\psi\vec{c}) = \nabla\psi\times\vec{c} + \psi\,\nabla\times\vec{c}$ (front book)
with $\psi = f(r)$ and $\vec{c} = \vec{n}\times\vec{p}$

$$\nabla\times\left[(\vec{n}\times\vec{p})\, f(r)\right] = \nabla f(r)\times(\vec{n}\times\vec{p}) + f(r)\,\nabla\times(\vec{n}\times\vec{p})$$

Since f depends only on r, then $\nabla f(r) = \underbrace{\hat{e}_r}_{\vec{n}}\frac{\partial f}{\partial r}$

and $\nabla f(r)\times(\vec{n}\times\vec{p}) = \left(\frac{\partial f}{\partial r}\right)\left[\vec{n}\times(\vec{n}\times\vec{p})\right]$. (1)

Consider now $\nabla\times(\vec{n}\times\vec{p}) =$

$= \vec{n}(\nabla\cdot\vec{p}) - \vec{p}(\nabla\cdot\vec{n}) + (\vec{p}\cdot\nabla)\vec{n} - (\vec{n}\cdot\nabla)\vec{p}$

From formula in front book

But $\nabla\cdot\vec{p} = 0$
and $(\vec{n}\cdot\nabla)\vec{p} = 0$ as well
since $\vec{p}$ is indep. of $\vec{x}$.

Then: $\nabla\times(\vec{n}\times\vec{p}) = -\vec{p}\,(\nabla\cdot\vec{n}) + (\vec{p}\cdot\nabla)\,\vec{n}$ (2)

$\nabla\cdot\vec{n} = \nabla\cdot\left(\frac{\vec{X}}{r}\right) = \frac{2}{r}$; $(\vec{p}\cdot\nabla)\vec{n} = \frac{1}{r}\left[\vec{p} - \vec{n}(\vec{p}\cdot\vec{n})\right]$

front Cover

front Cover

$\nabla\cdot\vec{n} f'(r)$ $= \frac{2}{r}f' + \frac{\partial f'}{\partial r} = \frac{2}{r}$ but $f' = 1$	"$(\vec{a}\cdot\nabla)\vec{n} f(r)$" with $f(r) = 1$ $\vec{a} = \vec{p}$

Then:

$$\nabla \times \left[(\vec{n} \times \vec{p}) f(r) \right] = \overset{1}{\left(\frac{\partial f}{\partial r}\right) \left[\vec{n} \times (\vec{n} \times \vec{p}) \right]}$$
$$+ f(r) \underbrace{\left[-\vec{p}\,\frac{2}{r} + \overset{2}{\frac{1}{r}\left[\vec{p} - (\vec{p} \cdot \vec{n}) \vec{n} \right]} \right]}_{-\frac{\vec{p}}{r} - \frac{\vec{n}}{r}(\vec{p} \cdot \vec{n})}$$

$$\frac{\partial f}{\partial r} = \frac{\partial}{\partial r}\left[\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right) \right] = \frac{\partial}{\partial r}\left[e^{ikr}\left(\frac{1}{r} - \frac{1}{ikr^2}\right) \right]$$

$$= ik\, e^{ikr}\left(\frac{1}{r} - \frac{1}{ikr^2}\right) + e^{ikr}\left(-\frac{1}{r^2} + \frac{2}{ikr^3}\right)$$

$$= \frac{e^{ikr}}{r}\left(ik - \frac{1}{r} - \frac{1}{r} + \frac{2}{ikr^2} \right)$$

$$= \frac{e^{ikr}}{r}\left(ik - \frac{2}{r} + \frac{2}{ikr^2} \right)$$

$$\frac{\partial f}{\partial r}\left[\vec{n} \times (\vec{n} \times \vec{p}) \right] = \frac{e^{ikr}}{r}\, ik \left[\vec{n} \times (\vec{n} \times \vec{p}) \right]$$
$$+ \frac{e^{ikr}}{r}\left(-\frac{2}{r} + \frac{2}{ikr^2} \right) \underbrace{\left[\vec{n} \times (\vec{n} \times \vec{p}) \right]}$$

From $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c}$

$$\vec{n} \times (\vec{n} \times \vec{p}) = (\vec{n} \cdot \vec{p})\, \vec{n} - \underbrace{(\vec{n} \cdot \vec{n})}_{=1} \vec{p}$$

Then:

$$\nabla \times \left[(\vec{n}\times\vec{p})f(r)\right] = \frac{e^{ikr}}{r}\, ik \left[\vec{n}\times(\vec{n}\times\vec{p})\right] + \frac{e^{ikr}}{r}\left(-\frac{2}{r}+\frac{2}{ikr^2}\right)\left[(\vec{n}\cdot\vec{p})\vec{n}-\vec{p}\right] \quad \Big\}\ 1$$

$$+ \frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\left[-\frac{2}{r}\vec{p}+\frac{\vec{p}}{r}-\frac{1}{r}(\vec{p}\cdot\vec{n})\vec{n}\right] \quad \Big\}\ 2$$

$$= \frac{e^{ikr}}{r}\, ik\, \underbrace{\left[\vec{n}\times(\vec{n}\times\vec{p})\right]}_{-(\vec{n}\times\vec{p})\times\vec{n}} + (\vec{n}\cdot\vec{p})\vec{n}\,\frac{e^{ikr}}{r}\underbrace{\left(-\frac{2}{r}+\frac{2}{ikr^2}-\frac{1}{r}\left(1-\frac{1}{ikr}\right)\right)}_{\left(-\frac{3}{r}+\frac{3}{ikr^2}\right)}$$

$$+\vec{p}\,\frac{e^{ikr}}{r}\underbrace{\left[\left(\frac{2}{r}-\frac{2}{ikr^2}\right)-\frac{1}{r}\left(1-\frac{1}{ikr}\right)\right]}_{\left(\frac{1}{r}-\frac{1}{ikr^2}\right)}$$

(i) the "$\vec{p}$" term is:

$$\frac{iZ_0 ck}{4\pi}\,\frac{e^{ikr}}{r}\left(\frac{1}{r}-\frac{1}{ikr^2}\right) = \sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{c}{4\pi}\left(\frac{ik}{r^2}-\frac{1}{r^3}\right)e^{ikr}$$

$$= \underbrace{\sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{1}{\sqrt{\mu_0\epsilon_0}}}_{\frac{1}{\epsilon_0}}\,\frac{1}{4\pi}\,e^{ikr}(-)\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)$$

$$= -\frac{1}{4\pi\epsilon_0}\,e^{ikr}\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)$$

✓ Correct compared with (9.18)

(ii) The "$(\vec{n}\times\vec{p})\times\vec{n}$" term is:

$$i\frac{Z_0 ck}{4\pi}\cdot(-)\,e^{ikr}\frac{ik}{r} = \underbrace{\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{c}{4\pi}}_{\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{1}{\sqrt{\mu_0\epsilon_0}}\frac{1}{4\pi}=\frac{1}{4\pi\epsilon_0}} k^2\frac{e^{ikr}}{r}$$

$$= \frac{1}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}$$

✓ Correct compared with (9.18)

(iii) The "$(\vec{p}.\vec{n})\vec{n}$" term is:

$$i\frac{Z_0 ck}{4\pi}\frac{e^{ikr}}{r}\left(-\frac{3}{r}+\frac{3}{ikr^2}\right) = \underbrace{\frac{Z_0 c}{4\pi}}_{\frac{1}{4\pi\epsilon_0}} e^{ikr}\,3\left(-\frac{ik}{r^2}+\frac{1}{r^3}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\,3\,e^{ikr}\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)$$

✓ Correct compared with (9.18)

Then:

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\vec{n}\times\vec{p})\times\vec{n}\,\frac{e^{ikr}}{r} + [3\vec{n}(\vec{n}.\vec{p})-\vec{p}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}}$$

(9.18)

units certainly be same, since $[k]=\frac{1}{L}$, but $\frac{1}{r^\alpha}$ dependence different.

(a)

In the radiation zone, we keep only $\frac{1}{r}$ terms:

From (9.18) dropping $\frac{1}{r^2}$.

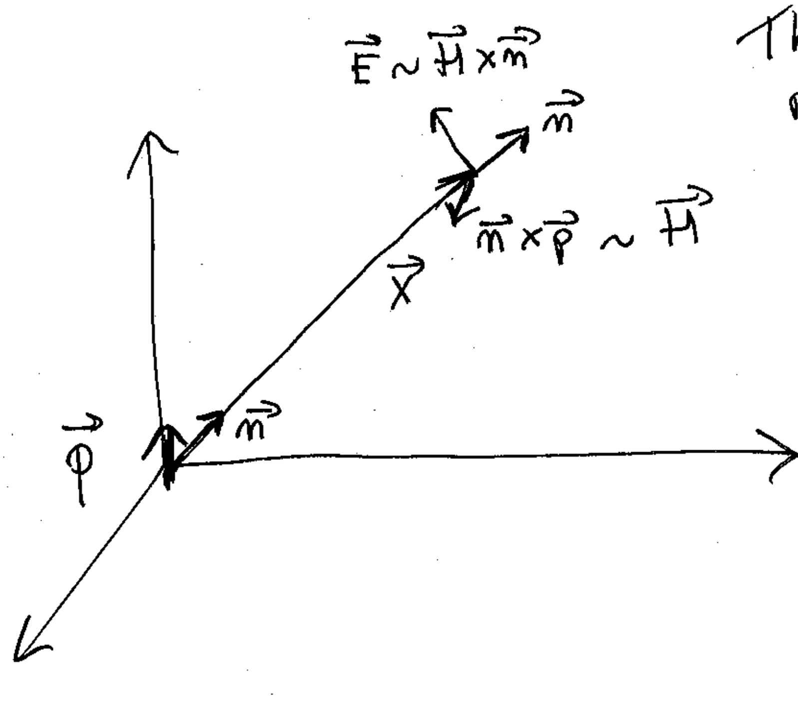
$$\vec{H} \cong \frac{ck^2}{4\pi}(\vec{n}\times\vec{p})\,\frac{e^{ikr}}{r} \qquad (9.19)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\,k^2\,(\vec{n}\times\vec{p})\times\vec{n}\,\frac{e^{ikr}}{r}$$

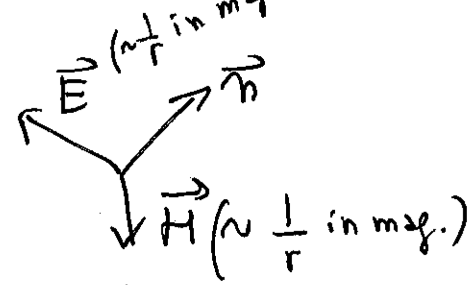
keep $\frac{1}{r}$ only in (9.18)

$$= \underbrace{\frac{1}{\epsilon_0 c}}_{\frac{\sqrt{\epsilon_0\mu_0}}{\epsilon_0}=\sqrt{\frac{\mu_0}{\epsilon_0}}\equiv Z_0}\cdot\underbrace{\frac{ck^2}{4\pi}\frac{e^{ikr}}{r}(\vec{n}\times\vec{p})}_{\vec{H}}\times\vec{n}$$

$$= Z_0(\vec{H}\times\vec{n})\quad\left(=\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{\mu_0\vec{H}}{\mu_0}\times\vec{n}\right)=\frac{1}{\sqrt{\mu_0\epsilon_0}}(\vec{B}\times\vec{n}) = c(\vec{B}\times\vec{n})\right)$$



Thus, we get a situation resembling plane waves as those studied in Chapter 7:



These are typical "radiation fields"

(b) In the other limit $r \ll \lambda$ we keep the $1/r^3$ term in $\vec{E}$ and $\frac{1}{r^2}$ in $\vec{H}$:

(and $e^{ikr} = e^{i\frac{2\pi r}{\lambda}} \sim 1$)

$$\vec{H} \cong \frac{ck^2}{4\pi}(\vec{n}\times\vec{p})\left(-\frac{1}{ikr^2}\right)$$

$$= \frac{i\overset{\omega}{ck}}{4\pi}(\vec{n}\times\vec{p})\frac{1}{r^2} = \boxed{\frac{i\omega}{4\pi}(\vec{n}\times\vec{p})\frac{1}{r^2}} \quad (9.20)$$

$$\boxed{\vec{E} \cong \frac{1}{4\pi\epsilon_0}\left(3\vec{n}(\vec{n}\cdot\vec{p})-\vec{p}\right)\frac{1}{r^3}}$$

just $\frac{1}{r^3}$

(exactly (4.13) found in Electrostatics)

$\vec{E}$ dominates over $\vec{H}$ because one has $\frac{1}{r^3}$ dependence and the other $\frac{1}{\lambda r^2}$.

$\vec{H}$ has ω and $1/r^2$, and $\omega \sim \frac{1}{\lambda}$, thus $\vec{H} \sim \frac{1}{\lambda r^2}$ and $\lambda \gg r$ in near zone, thus $\vec{E}$ dominates.

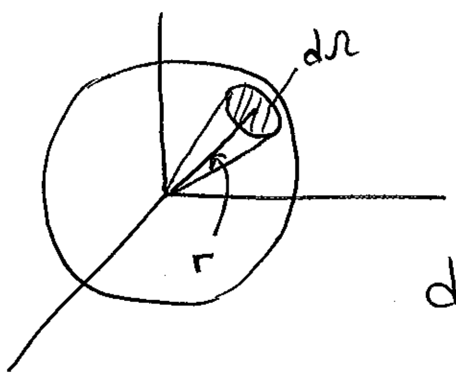
<u>The fields in the "near zone" are predominantly electric.</u>

$\vec{H}$ vanishes if $\omega \to 0$, because it is induced by Faraday's law.

$\vec{E}$ does not vanish if $\vec{p}$ is $\neq 0$, ie if $\rho(\vec{x}')$ is not zero at all $\vec{x}'$.

$\vec{S} = \vec{E} \times \vec{H}$, the Poynting vector, represents energy flow. We are interested in the power radiated from the "center" where the localized charges is. Thus, we want the projection of $\vec{S}$ radial (away from center):

$$\vec{n} \cdot (\vec{E} \times \vec{H})$$



Energy flow through shaded region (dP) is

$$dP = \underbrace{r^2 d\Omega}_{\text{area shaded region}} \; \vec{n} \cdot (\vec{E} \times \vec{H})$$

Ch6: Total energy flow $\oint \vec{S} \cdot \vec{n}\, da$. / A (per unit time!)

To take the "time average" the recipe is in page 264 (paragraphs "For time averages...")

Take "$\frac{1}{2} \mathrm{Re}(A.B^*)$"

$$\boxed{\frac{dP}{d\Omega} \overset{\text{time averaged}}{=} \frac{1}{2} \mathrm{Re}\left[r^2 \vec{n} \cdot (\vec{E} \times \vec{H}^*)\right]} \qquad (9.21)$$

Using the radiation zone fields:

$$\vec{H} = \frac{ck^2}{4\pi} (\vec{n} \times \vec{p}) \frac{e^{ikr}}{r}$$

$$\vec{E} = Z_0 (\vec{H} \times \vec{n})$$

$$\vec{E}\times\vec{H}^*\cdot\vec{n} = Z_0\left[(\vec{H}\times\vec{n})\times(\vec{n}\times\vec{p})\ \frac{ck^2}{4\pi}\frac{e^{-ikr}}{r}\right]\cdot\vec{n}$$

$$= Z_0\left(\frac{ck^2}{4\pi}\right)^2\frac{e^{ikr}}{r}\,\frac{e^{-ikr}}{r}\ \underbrace{\left[((\vec{n}\times\vec{p})\times\vec{n})\times(\vec{n}\times\vec{p})\right]\cdot\vec{n}}$$

$$= \frac{Z_0\,c^2k^4}{(4\pi)^2\,r^2}\ \underbrace{\underbrace{\left[(\vec{n}\times\vec{p})\times\vec{n}\right]}_{\text{"}\vec{b}\text{"}}\cdot\underbrace{\left[(\vec{n}\times\vec{p})}_{\vec{c}}\times\underbrace{\vec{n}}_{\vec{a}}\right]}_{\left|\underbrace{(\vec{n}\times\vec{p})\times\vec{n}}_{\text{direction of the electric field}}\right|^2}$$

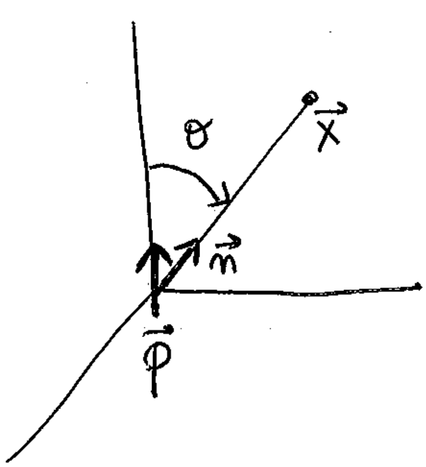
We use

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

$\vec{a} = \vec{n}$, $\vec{b} = (\vec{n}\times\vec{p})\times\vec{n}$, $\vec{c} = \vec{n}\times\vec{p}$

Thus:

$$\frac{1}{2}\,\mathrm{Re}\left[r^2\,\vec{n}\cdot(\vec{E}\times\vec{H}^*)\right] = \boxed{\frac{1}{2}\,\frac{Z_0c^2k^4}{(4\pi)^2}\left|(\vec{n}\times\vec{p})\times\vec{n}\right|^2 = \frac{dP}{d\Omega}} \qquad (9.22)$$

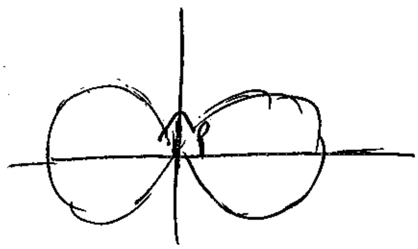


$$\vec{n}\times\vec{p} = \underbrace{|\vec{n}||\vec{p}|\sin\theta}\ \underbrace{\hat{e}_u}_{\text{along } \vec{n}\times\vec{p}}$$

$$\left|(\vec{n}\times\vec{p})\times\vec{n}\right| = \underbrace{|\vec{n}|}_{=1}|\vec{p}|\sin\theta\ \underbrace{|\vec{n}|}_{=1}\underbrace{|\hat{e}_u|}_{=1}\sin\varphi$$

angle between $\vec{n}$ and $\vec{n}\times\vec{p}$ which is $\pi/2$

$$\boxed{\frac{dP}{d\Omega} = \frac{Z_0 c^2 k^4}{32\pi^2} |\vec{p}|^2 \sin^2\theta} \qquad (9.23)$$



Integrating in $d\Omega$:

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \sin^2\theta \, d\theta = \cancel{\left.\frac{\sin^4\theta}{4}\right|_0^{\pi}}$$

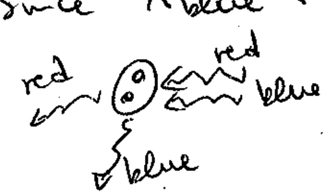
$$= \left(\int_0^{\pi} \sin\theta \, d\theta - \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta\right) 2\pi$$

$$= 2\pi\left(-\cos\theta\Big|_0^{\pi} + \frac{\cos^3\theta}{3}\Big|_0^{\pi}\right) = 2\pi\left(2 - \frac{2}{3}\right) = \frac{4}{3}\cdot 2\pi$$

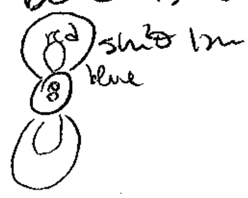
$$P = \frac{Z_0 c^2 k^4}{32\pi^2} \, 2\pi \frac{4}{3} |\vec{p}|^2$$

$$= \boxed{\frac{Z_0 c^2 k^4}{12\pi} \cdot |\vec{p}|^2}. \qquad (9.24)$$

Total power radiated. Since $k = \frac{\omega}{c}$, then $P \propto \omega^4$. As explained in Griffiths, this accounts for the blue color of the sky. Sunlight estimulates atoms of the atmosphere to oscillate as tiny dipoles. This effect is stronger, i.e. the power radiated is stronger, for ω large. Since large ω means small λ and since $\lambda_{blue} < \lambda_{red}$, then blue is the dominant color.

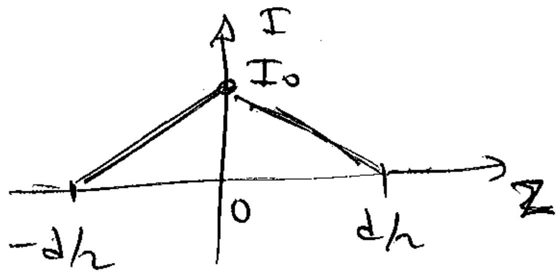


or



Example:

Antenna oriented along z-axis of length "d" with a narrow gap at the center. The current in each half is in the same direction with a value I_0 at the center and falling linearly to the ends.



There is no discussion on how this is achieved in practice.

$$\boxed{I(z) = I_0\left(1 - \frac{|z|}{d/2}\right)} \qquad (9.25)$$

and with a $e^{-i\omega t}$ time dependence.

From $\rho = \frac{1}{i\omega}\nabla\cdot\vec{J} = \frac{I_0}{i\omega}\frac{d}{dz}\left(1 - \frac{2|z|}{d}\right)$

(9.15)

$$\rho(z>0) = \frac{I_0}{i\omega}\left(\frac{-2}{d}\right)\frac{dz}{dz} = -\frac{2I_0}{i\omega d} = \frac{2iI_0}{\omega d}$$

$$\rho(z<0) = \frac{I_0}{i\omega}\left(\frac{-2}{d}\right)\frac{d(-z)}{dz} = \frac{-2iI_0}{\omega d}$$

$$\boxed{\rho(z) = \pm\frac{2iI_0}{\omega d}} \qquad \left(\begin{array}{l}+ \text{ for } z>0 \\ - \text{ for } z<0\end{array}\right) \rightarrow$$

It is indeed like a dipole. The $e^{-i\omega t}$ makes the sign in each half to change from + to - and viceversa.

↑ Real part of $\rho(z)e^{-i\omega t}$ is what matters, of course.

$$\underset{\substack{\uparrow \\ \text{dipole moment along } z \text{ axis.}}}{p_z} = \int_{-d/2}^{d/2} z\,\rho(z)\,dz = \frac{2 i I_0}{\omega d}\Bigg(\underbrace{\int_0^{d/2} z\,dz}_{\frac{z^2}{2}\Big|_0^{d/2} = \frac{1}{2}\frac{d^2}{4}} - \underbrace{\int_{-d/2}^{0} z\,dz}_{-\frac{1}{2}\frac{d^2}{4}}\Bigg)$$

$$= \frac{2 i I_0}{\omega d}\frac{d^2}{4} = \boxed{\frac{i I_0 d}{2\omega} = p} \quad (9.27)$$

Then
$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{32\pi^2}\underbrace{\frac{I_0^2 d^2}{4\omega^2}}_{|p|^2}\sin^2\theta \qquad \omega^2 \to k^2c^2$$

$$= \boxed{\frac{Z_0 I_0^2}{128\pi^2}(kd)^2\sin^2\theta} \quad (9.28)$$

Total power:

$$P = \frac{Z_0 I_0^2 (kd)^2}{128\pi^2}\, 2\pi \cdot \underbrace{\frac{4}{3}}_{\int \text{ already done a couple of pages before.}} = \boxed{\frac{Z_0 I_0^2 (kd)^2}{48\,\pi}} \quad (9.29)$$

Note: if we would have kept other powers of $1/r$ in $\vec{E}$ and $\vec{H}$ (such as $1/r^2, 1/r^3, \ldots$) their contribution would have been cancelled when integrating. So "radiation fields" are those that transport energy out to infinity and only $1/r$ can do it.