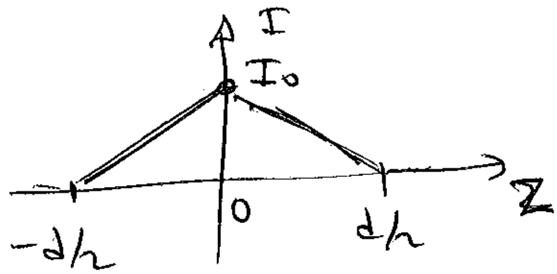
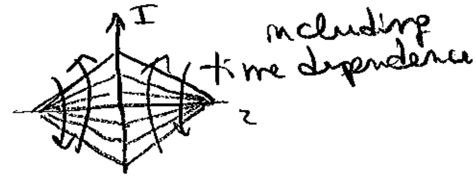
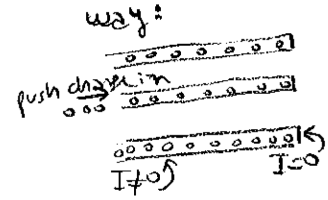
Example:

Antenna oriented along z-axis of length "d" with a narrow gap at the center. The current in each half is in the same direction with a value I_0 at the center and falling linearly to the ends.





There is no discussion on how this is achieved in practice. But injecting/or removing more charge at one end is a way:



$$\boxed{I(z) = I_0\left(1 - \frac{|z|}{d/2}\right)} \qquad (9.25)$$

and with a $e^{-i\omega t}$ time dependence.

From $\rho = \frac{1}{i\omega}\nabla\cdot\vec{J} = \frac{I_0}{i\omega}\frac{d}{dz}\left(1 - \frac{2|z|}{d}\right)$

(9.15)

$$\rho(z>0) = \frac{I_0}{i\omega}\left(\frac{-2}{d}\right)\frac{dz}{dz} = -\frac{2I_0}{i\omega d} = \frac{2iI_0}{\omega d}$$

$$\rho(z<0) = \frac{I_0}{i\omega}\left(\frac{-2}{d}\right)\frac{d(-z)}{dz} = -\frac{2iI_0}{\omega d}$$

$$\boxed{\rho(z) = \pm\frac{2iI_0}{\omega d}} \qquad \left(\begin{array}{l} + \text{ for } z>0 \\ - \text{ for } z<0 \end{array}\right) \rightarrow$$

It is indeed like a dipole. The $e^{-i\omega t}$ makes the sign in each half to change from + to − and viceversa.

Real part of $\rho(z)e^{-i\omega t}$ is what matters, of course.

$\mathrm{Re}\,\rho = \pm\frac{2I_0}{\omega d}\sin(\omega t)$

$$P_z = \int_{-d/2}^{d/2} z\,\rho(z)\,dz = \frac{2iI_0}{\omega d}\Bigg(\underbrace{\int_0^{d/2} z\,dz}_{\frac{z^2}{2}\Big|_0^{d/2} = \frac{1}{2}\frac{d^2}{4}} - \underbrace{\int_{-d/2}^{0} z\,dz}_{-\frac{1}{2}\frac{d^2}{4}}\Bigg)$$

dipole moment along z axis.

$$= \frac{2iI_0}{\omega d}\,\frac{d^2}{4} = \boxed{\frac{cI_0 d}{2\omega} = p} \quad (9.27)$$

Then
$$\frac{dP}{d\Omega} = \frac{c^2 \overset{\sqrt{\mu_0/\epsilon_0}}{Z_0}\, k^4}{32\pi^2}\cdot \underbrace{\frac{I_0^2 d^2}{4\omega^2}}_{|p|^2}\cdot \sin^2\theta \qquad \omega^2 \to k^2c^2$$

$$= \boxed{\frac{Z_0 I_0^2}{128\pi^2}(kd)^2 \sin^2\theta} \quad (9.28)$$

Total power:

$$P = \frac{Z_0 I_0^2 (kd)^2}{128\pi^2}\cdot 2\pi\cdot \underbrace{\frac{4}{3}}_{\int \text{ already done a couple of pages before.}} = \boxed{\frac{Z_0 I_0^2 (kd)^2}{48\,\pi}} \quad (9.29)$$

From $P = \frac{RI^2}{2}$, the comb[ination] $\frac{Z_0 (kd)^2}{48\pi}$ plays the role of a "radiation resistance" even if conductivity is perfect.

Note: if we would have kept other powers of $1/r$ in $\vec{E}$ and $\vec{H}$ (such as $1/r^2, 1/r^3, \ldots$) their contribution would have been cancelled when integrating. So "radiation fields" are those that transport energy out to infinity and only $1/r$ can do it.

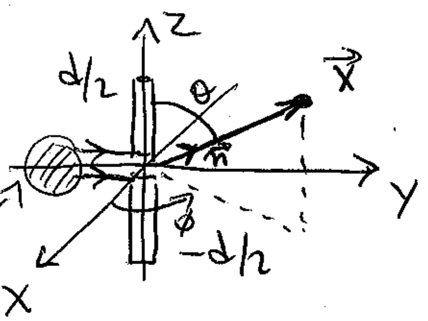
9.4 Linear Antenna

For some simple cases we can directly use the formula

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\,\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,d^3x \qquad (9.3)$$

without having to expand in near, far, etc. and/or dipoles, quadrupoles, etc.

Consider the case shown in the figure



Current assumed sinusoidal in time and space with $k = \omega c$, symmetric in the two arms.

The current is 0 at the ends

Small gap at center

$$\vec{J}(\vec{x}) = I \sin\left(\frac{kd}{2} - k|z|\right)\delta(x)\,\delta(y)\,\hat{e}_z$$

and $|z| < d/2$

Similar to case

solved before.

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} I \hat{e}_z \int_{-d/2}^{+d/2} dz' \sin\left(\frac{kd}{2} - k|z'|\right) e^{ik|\vec{x}-\vec{x}'|}$$

If we now assume the "radiation zone" limit then:

$$|\vec{x}-\vec{x}'| \approx r - \vec{n}\cdot\vec{x}' \qquad (9.7)$$

Then

$$e^{ik|\vec{x}-\vec{x}'|} \approx e^{ikr}\, e^{-ik\underbrace{\vec{n}\cdot\vec{x}'}}$$

But $\vec{x}'$ only points along z i.e. $\vec{x}' = z'\hat{e}_z$ and $\vec{n} = \frac{1}{r}[r\cos\theta\, \hat{e}_z + \ldots]$

Then $\vec{n}\cdot\vec{x}' = z'\cos\theta$

$$\boxed{\underset{\text{radiation zone}}{\vec{A}(\vec{x}) \approx} \hat{e}_z \frac{\mu_0 I}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{+d/2} dz' \sin\left(\frac{kd}{2} - k|z'|\right) e^{-ikz'\cos\theta}} \quad (9.54)$$

From $|\vec{x}-\vec{x}'|$ in denominator

see integral in next page.

$$= \hat{e}_z \frac{\mu_0 I}{4\pi} \cdot \frac{e^{ikr}}{r} \cdot \frac{2}{k\sin^2\theta} \left[\cos\left(\frac{kd\cos\theta}{2}\right) - \cos\left(\frac{kd}{2}\right)\right] \qquad (9.55)$$

no dipole approx.

Note the "more complicated" θ dependence than in the dipole, quadruple, expansion.

It will not be shown explicitly but we will simply accept the result of the author that:

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\right|^2$$

In principle: get $\vec{H}$, then $\vec{E}$, then $\frac{dP}{d\Omega}$.

If $kd = \pi$ we get $\frac{Z_0 I^2}{8\pi^2}\left|\frac{\cos\left(\frac{\pi}{2}\cos\theta\right) - \overbrace{\cos(\pi/2)}^{0}}{\sin\theta}\right|^2$

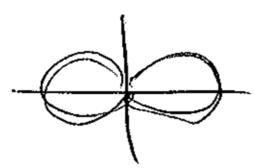
For $\theta = \pi/2$, $\frac{\cos(\pi/2 \cos\theta)}{\sin\theta} = \frac{\cos(\pi/2 \cdot 0)}{1} = \frac{1}{1}$.

For $\theta = 0$, $\frac{\cos\left(\frac{\pi}{2}\cdot 1\right)}{\sin 0} = \frac{0}{0}$.

$$\lim_{\theta\to 0}\frac{\cos(\pi/2\cos\theta)}{\sin\theta} = \lim_{\theta\to 0}\frac{(-)\sin(\pi/2\cos\theta)\ \pi/2\,(-\sin\theta)}{\cos\theta} = 0$$

L'Hopital

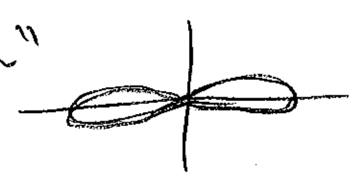
Then, this is of the form



Very similar to a single dipole pattern.

[By a combination of basic antennas of this form and by adjusting the phases of the currents in each arbitrary radiation pattern can be obtained]

For $kd = 2\pi$, it is a bit "thinner"



Integral needed to deduce (9.55)

$$I_1 = \int_0^{d/2} dz \, \sin\left(\frac{kd}{2} - kz\right) e^{-ikz\cos\theta} =$$

$$= \int_0^{d/2} dz \; \frac{e^{i\left(\frac{kd}{2} - kz\right)} - e^{-i\left(\frac{kd}{2} - kz\right)}}{2i} \cdot e^{-ikz\cos\theta}$$

$$= \frac{e^{\frac{ikd}{2}}}{2i} \underbrace{\int_0^{d/2} dz \; e^{-ikz(1+\cos\theta)}}_{\left.\frac{e^{-ikz(1+\cos\theta)}}{-ik(1+\cos\theta)}\right|_0^{d/2}} - \frac{e^{-\frac{ikd}{2}}}{2i} \underbrace{\int_0^{d/2} dz \; e^{ikz(1-\cos\theta)}}_{\left.\frac{e^{ikz(1-\cos\theta)}}{ik(1-\cos\theta)}\right|_0^{d/2}}$$

$$= \frac{e^{\frac{ikd}{2}}}{2i} \frac{\left(e^{-\frac{ikd}{2}} e^{-\frac{ikd}{2}\cos\theta} - 1\right)}{(-ik(1+\cos\theta))} - \frac{e^{-\frac{ikd}{2}}}{2i} \frac{\left(e^{\frac{ikd}{2}(1-\cos\theta)} - 1\right)}{ik(1-\cos\theta)}$$

$$= \frac{e^{-\frac{ikd}{2}\cos\theta}}{2k(1+\cos\theta)} - \frac{e^{\frac{ikd}{2}}}{2k(1+\cos\theta)} - \frac{e^{-\frac{ikd}{2}\cos\theta}}{(-2k(1-\cos\theta))} + \frac{e^{-\frac{ikd}{2}}}{(-2k(1-\cos\theta))}$$

$$I_2 = \int_{-d/2}^{0} dz \, \sin\left(\frac{kd}{2} + kz\right) e^{-ikz\cos\theta} \underset{z=-u}{\overset{\uparrow}{=}}$$

$$= \int_{d/2}^{0} (-du) \, \sin\left(\frac{kd}{2} - ku\right) e^{+iku\cos\theta}$$

$$= \int_0^{d/2} du \, \sin\left(\frac{kd}{2} - ku\right) e^{iku\cos\theta} = I_1(-\cos\theta)$$

$$I_1 = e^{-ik\frac{d}{2}\cos\theta}\left[\frac{1}{2k(1+\cos\theta)} + \frac{1}{2k(1-\cos\theta)}\right]$$

$$- \frac{e^{+\frac{ikd}{2}}}{2k(1+\cos\theta)} - \frac{e^{-\frac{ikd}{2}}}{2k(1-\cos\theta)}$$

$$I_2 = e^{\frac{ikd}{2}\cos\theta}\left[\frac{1}{2k(1-\cos\theta)} + \frac{1}{2k(1+\cos\theta)}\right]$$

$$- \frac{e^{\frac{ikd}{2}}}{2k(1-\cos\theta)} - \frac{e^{-\frac{ikd}{2}}}{2k(1+\cos\theta)}$$

$$\frac{1}{2k(1+\cos\theta)} + \frac{1}{2k(1-\cos\theta)} = \frac{2k(1-\cos\theta) + 2k(1+\cos\theta)}{4k^2(1+\cos\theta)(1-\cos\theta)} =$$

$$= \frac{4k}{4k^2\underbrace{(1-\cos^2\theta)}_{\sin^2\theta}} = \frac{1}{k\sin^2\theta}$$

$$I_1+I_2 = \frac{1}{k\sin^2\theta}\underbrace{\left(e^{-\frac{ikd}{2}\cos\theta} + e^{\frac{ikd}{2}\cos\theta}\right)}_{2\cos\left(\frac{kd}{2}\cos\theta\right)} \underbrace{- \frac{e^{\frac{ikd}{2}}}{k\sin^2\theta} - \frac{e^{-\frac{ikd}{2}}}{k\sin^2\theta}}_{-\frac{2}{k\sin^2\theta}\cos\left(\frac{kd}{2}\right)}$$

$$I_1+I_2 = \frac{2}{k\sin^2\theta}\left[\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)\right]$$