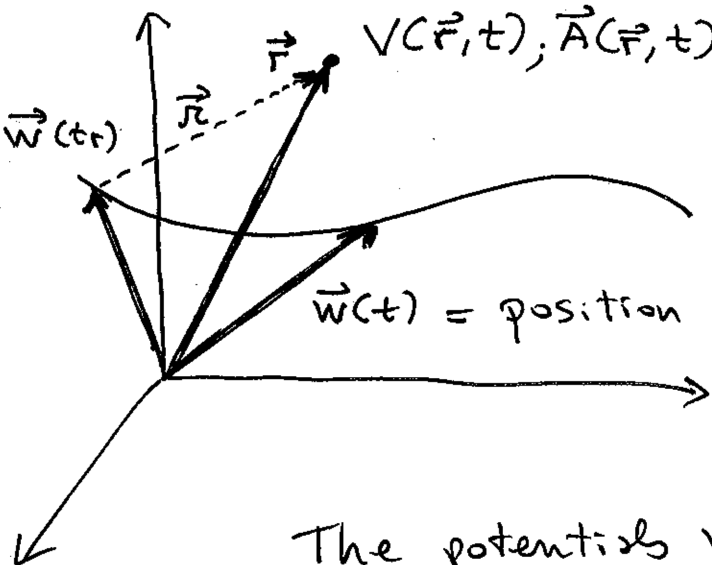
Radiation from a Point charge

Liénard – Wiechert Potentials

(Sec. 9.2.1 second ed.)

We want to calculate the $\vec{E}$ and $\vec{B}$ fields produced by a charge that is moving (and accelerating). In particular, the radiation fields.



$\vec{w}(t)$ = position of $\underline{\underline{q}}$ at time $\underline{\underline{t}}$ (trajectory)

The potentials $V(\vec{r},t), \vec{A}(\vec{r},t)$ depend on the status of the charge at time $\underline{\underline{t_r}}$, which is determined via

$$|\vec{r} - \vec{w}(t_r)| = c(t - t_r)$$

"retarded position of the charge"; $\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$

Now consider many particles:

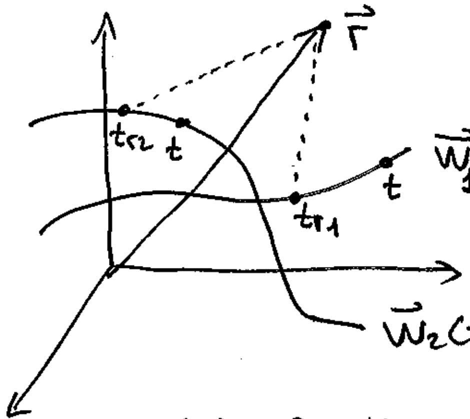
$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)\,d\tau}{|\vec{r}-\vec{w}(t_r)|}$$

retarded potential

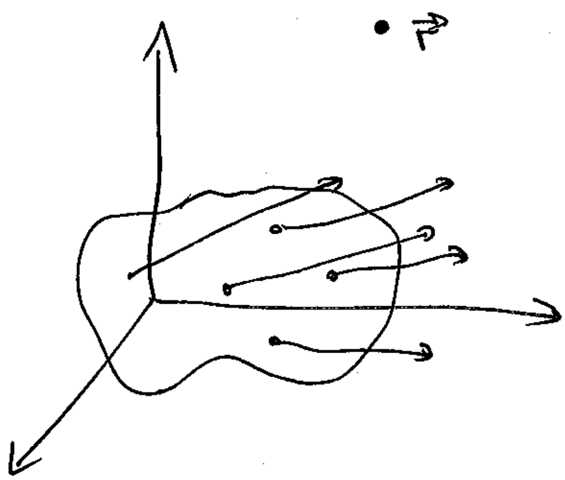
[$d\vec{x}'$ in the notation of Jackson; $\vec{r}$ is $\vec{x}$ in the notation of Jackson.]

The most important point to realize is that "t_r" will be different for each point $\vec{r}'$! Thus, t_r is not a unique time.

For instance, imagine that the "ρ" is made out of two charges:



At $\vec{r}$, the potential will have two components: the contribution of particle 1 that occurs at t_{r1} and the contribution of particle 2 that occurs at t_{r2}. For a full "body", each point has its own trajectory, making the problem complicated.



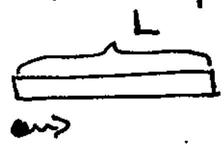
Moreover, remember that objects modify their lengths if "c" and their velocity are comparable.

And, to complicate matters worse, a point-like charge has to be considered as the limit of a distribution ρ.

What is the length of a moving object when seen at the distance?

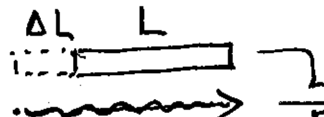
The answer will be based on geometry, not on relativity, although the formulas look similar.

Consider a train. At time $t=0$, from the caboose light is emitted forward.



$t=0$

When is the front of the train and the light emitted from the caboose at the same place?



→ The distance traveled by light emitted by caboose is $L+\Delta L$, that we call L'. The time it took is $\frac{L'}{c}$.

→ The distance traveled by front of train is $\Delta L = L'-L$, the velocity is v, and the time is: $\frac{L'-L}{v}$

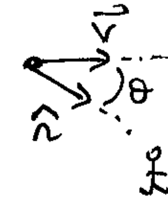
Thus: $\frac{L'-L}{v} = \frac{L'}{c} \implies L' = \frac{L}{1-\frac{v}{c}}$

The train looks longer than it is, if train goes into observer's direction.

Repeating for a velocity going the other way, i.e. away from observer, then $L' = \frac{L}{1+\frac{v}{c}}$ and the train looks shorter.

In general: it can be shown that

$$\boxed{L' = \frac{L}{1 - \frac{\hat{n}\cdot\vec{v}}{c}}}$$



where $\hat{n}$ = unit vector from train to observer.

Example: for the case above $\vec{v}$, $\hat{n}$, $\hat{n}\cdot\vec{v} = 1$, observer

and we get back $L' = \frac{L}{1-\frac{v}{c}}$.

The only distance affected is the one along $\vec{v}$, the other two are untouched. Then, as a volume we can say:

$$V'_{ol} = \frac{V_{ol}}{1 - \frac{\hat{\imath}\cdot\vec{v}}{c}}.$$

Now let us go back to E&M. We had the integral $\int \frac{\rho(\vec{r}', t_r)\, d\tau}{|\vec{r} - \vec{w}(t_r)|}$ and we have to use the information about length expansion or contraction.

The apparent volume $d\tau^{app}$ is related to $d\tau^{real}$ by the same type of factors: The "$d\tau$" in the integral is the apparent one, thus we have

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)\, d\tau}{\underbrace{|\vec{r} - \vec{w}(t_r)|}_{|\vec{\imath}|}} \; \underset{\substack{\uparrow \\ \text{one charge} \\ \text{moving at} \\ \text{speed } \vec{v}}}{=} \; \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{\imath}|} \frac{\overbrace{\int \rho(\vec{r}', t_r)\, d\tilde{\tau}}^{\to\, q}}{\left(1 - \frac{\hat{\imath}\cdot\vec{v}}{c}\right)}$$

$(d\tilde{\tau} = \text{real } d^3\vec{x}')$

In other words: $\int \rho(\vec{r}', t_r)\, d\tau$ is not q because it is not all points at the same time

$$\boxed{V(\vec{r}, t) = \frac{q}{|\vec{\imath}|} \cdot \frac{1}{\left(1 - \frac{\hat{\imath}\cdot\vec{v}}{c}\right)}}$$

(9.86 Griffiths)

So $\int \rho(\vec{r}', t_r)\, d\tau \underset{\substack{\uparrow \\ \text{Jackson's notation}}}{=} \int \rho(\vec{x}', t_r)\, d^3\vec{x}' \neq q$. It is like imagining the charge as made of little pieces (••••) Since the "apparent" length is, say, longer, it looks as if having more charge. This becomes clear in the following example:

Maybe the mass may have similar features.

Intuitively, let us consider the following: Suppose I have a system of length L with a uniformly distributed charge q. Then, its charge density is (q/L).

Now suppose this charge starts moving at velocity v. From the distance, the apparent length will be L'. Then, the "observer" will believe the charge is:

$$\underbrace{L'}_{\text{apparent length}} \cdot \underbrace{\left(\frac{q}{L}\right)}_{\text{density of charge.}}$$

This ~~$\left(\frac{q}{L}\right)$~~ $L'\left(\frac{q}{L}\right)$ is the equivalent of $\int \rho(\vec{r}', tr)\, d\tau$ ($d^3\vec{r}'$ in Jackson's notation) in the notation of Griffiths:

$$\int \rho(\vec{r}', tr)\, d\tau = L'\left(\frac{q}{L}\right)$$

To arrive to the correct result we need to multiply by $\left(\frac{L'}{L}\right)^{-1} = \frac{L}{L'}$ → Crucial: this ratio is indep. of L but only depends on v. Thus, it is there irrespective of size of charge, including a small particle.

$$\int \rho(\vec{r}', tr)\, d\tau \, \frac{L}{\underbrace{L'}_{1-\frac{v}{c}}} = L'\left(\frac{q}{L}\right)\frac{L}{L'} = q$$

(assume all movement is along same line i.e. drop $\cos\theta$)

Then: $$\boxed{\int_{L'} \rho(\vec{r}', tr)\, d\tau = \frac{q}{1-\frac{v}{c}}}$$

for a "small" charge such that $\frac{1}{|\vec{r}|}$ can be removed from integral.

Result valid even if $L', L \to 0$

In particular for a pointlike charge

The fields of a point charge in motion

We have to use the usual formulas

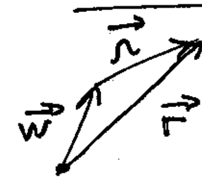
$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

($V = \Phi$ in Jackson's notation)

and $\vec{B} = \nabla \times \vec{A}$

(9.90)

where $V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{|\vec{\imath}|\left(1 - \frac{\hat{\imath}\cdot\vec{v}}{c}\right)} = \boxed{\frac{1}{4\pi\epsilon_0} \cdot \frac{qc}{(\imath c - \vec{\imath}\cdot\vec{v})}}$



(9.80) Griffiths

With regards to $\vec{A}(\vec{r}, t)$, it is easy to use $\rho \vec{v}$ as the current density of a rigid object:

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\rho(\vec{r}', t_r)}{|\vec{\imath}|} \vec{v}(t_r)\, d\tau \underset{\text{point charge}}{=} \frac{\mu_0}{4\pi} \frac{\overset{\text{velocity at the retarded time}}{\vec{v}}}{|\vec{\imath}|} \underbrace{\int \rho(\vec{r}', t_r)\, d\tau}_{\frac{q}{1 - \frac{\hat{\imath}\cdot\vec{v}}{c}}}$$

$$\vec{A}(\vec{r}, t) = \underbrace{\mu_0 \epsilon_0}_{1/c^2} \vec{v} \underbrace{\frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{\imath}|} \frac{q}{\left(1 - \frac{\hat{\imath}\cdot\vec{v}}{c}\right)}}_{V(\vec{r}, t)}$$

Liénard Wiechert potentials:

$$\boxed{\vec{A}(\vec{r}, t) = \frac{\vec{v}}{c^2} V(\vec{r}, t)} \qquad (9.87)$$

for a pointlike charge.

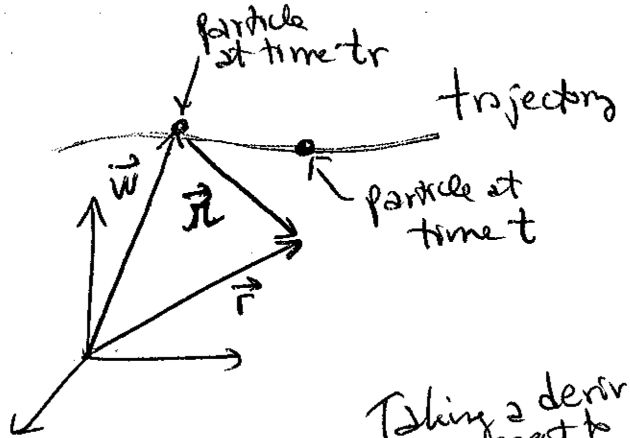
to get $\vec{E}$

We need to calculate ∇V where

$$V = \frac{1}{4\pi\epsilon_0} \cdot \frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})}$$

Note that $\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$

and $\vec{v} = \left.\frac{d\vec{w}}{dt}\right|_{t_r}$



Taking a deriv. with respect to $\vec{r}$ of a time t_r is $\neq 0$ because t_r is different at each $\vec{r}$.

$$\nabla V = \frac{qc}{4\pi\epsilon_0} \cdot \frac{(-1)}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^2} \cdot \nabla(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})$$

$$\nabla\left(\frac{1}{\varphi(\vec{r})}\right) = -\frac{1}{\varphi(\vec{r})^2} \cdot \nabla\varphi(\vec{r})$$

Obvious if you study ∇ component by component.

Note that $\nabla(\mathscr{r}c) = c\nabla\mathscr{r} = -c^2\nabla t_r$

$\mathscr{r} = c(t - t_r)$

With regards to $\nabla(\vec{\mathscr{r}}\cdot\vec{v})$ we do the following:

$$\nabla(\vec{\mathscr{r}}\cdot\vec{v}) = (\vec{\mathscr{r}}\cdot\nabla)\vec{v} + (\vec{v}\cdot\nabla)\vec{\mathscr{r}} + \vec{\mathscr{r}}\times(\nabla\times\vec{v}) + \vec{v}\times(\nabla\times\vec{\mathscr{r}})$$

Griffiths front cover

(a) Consider first $(\vec{\mathscr{r}}\cdot\nabla)\vec{v} = \left(\mathscr{r}_x\frac{\partial}{\partial x} + \mathscr{r}_y\frac{\partial}{\partial y} + \mathscr{r}_z\frac{\partial}{\partial z}\right)\vec{v}(t_r) =$

$$= \mathscr{r}_x \frac{\partial\vec{v}}{\partial t_r}\frac{\partial t_r}{\partial x} + \mathscr{r}_y \frac{\partial\vec{v}}{\partial t_r}\frac{\partial t_r}{\partial y} + \mathscr{r}_z \frac{\partial\vec{v}}{\partial t_r}\frac{\partial t_r}{\partial z} = \underbrace{\frac{\partial\vec{v}}{\partial t_r}}_{\text{acceleration } \vec{a} \text{ of the particle at the retarded time}}(\vec{\mathscr{r}}\cdot\nabla t_r)$$

$$= \vec{a}\,(\vec{\mathscr{r}}\cdot\nabla t_r) \qquad (9.94)$$

(b) $(\vec{v}\cdot\nabla)\vec{\mathscr{r}} = (\vec{v}\cdot\nabla)(\vec{r}-\vec{w}) = (\vec{v}\cdot\nabla)\vec{r} - (\vec{v}\cdot\nabla)\vec{w}$

$$(\vec{v}\cdot\nabla)\vec{r} = \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)\underbrace{\vec{r}}_{(x\hat{e}_x+y\hat{e}_y+z\hat{e}_z)} =$$

$$= v_x\hat{e}_x + v_y\hat{e}_y + v_z\hat{e}_z = \vec{v}. \qquad (9.96)$$

$$(\vec{v}\cdot\nabla)\vec{w} = \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)\vec{w} =$$

$$= v_x\frac{\partial\vec{w}}{\partial t_r}\cdot\frac{\partial t_r}{\partial x} + v_y\frac{\partial\vec{w}}{\partial t_r}\cdot\frac{\partial t_r}{\partial y} + v_z\frac{\partial\vec{w}}{\partial t_r}\cdot\frac{\partial t_r}{\partial z}$$

$$= \underbrace{\frac{\partial\vec{w}}{\partial t_r}}_{\vec{v}}(\vec{v}\cdot\nabla t_r) = \vec{v}\,(\vec{v}\cdot\nabla t_r)$$

at retarded time

(c) $$\nabla\times\vec{v} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \hat{e}_x\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{e}_y\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{e}_z\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

For each one I can do $\frac{\partial v_y}{\partial x} = \frac{\partial v_y}{\partial t_r}\frac{\partial t_r}{\partial x}$

$$= \hat{e}_x\left(\overbrace{\frac{\partial v_z}{\partial t_r}}^{a_z}\frac{\partial t_r}{\partial y} - \overbrace{\frac{\partial v_y}{\partial t_r}}^{a_y}\frac{\partial t_r}{\partial z}\right) + \hat{e}_y\left(\overbrace{\frac{\partial v_x}{\partial t_r}}^{a_x}\frac{\partial t_r}{\partial z} - \overbrace{\frac{\partial v_z}{\partial t_r}}^{a_z}\frac{\partial t_r}{\partial x}\right) + \hat{e}_z\left(\overbrace{\frac{\partial v_y}{\partial t_r}}^{a_y}\frac{\partial t_r}{\partial x} - \overbrace{\frac{\partial v_x}{\partial t_r}}^{a_x}\frac{\partial t_r}{\partial y}\right) = \underset{!}{-}\begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ a_x & a_y & a_z \\ \frac{\partial t_r}{\partial x} & \frac{\partial t_r}{\partial y} & \frac{\partial t_r}{\partial z} \end{vmatrix} = -(\vec{a}\times\nabla t_r)$$

$$(9.97)$$

(d) $$\nabla \times \vec{\mathscr{r}} = \underbrace{\nabla \times \vec{r}}_{=0} - \nabla \times \vec{w} = -(\vec{v} \times \nabla t_r) \qquad (9.99)$$

by exactly the same procedure as in (9.97)

Putting all together, (9.93) becomes:

$$\nabla(\vec{\mathscr{r}}\cdot\vec{v}) = \underset{(a)}{\vec{a}(\vec{\mathscr{r}}\cdot\nabla t_r)} + \underset{(b)}{\vec{v} - \vec{v}(\vec{v}\cdot\nabla t_r)}$$
$$\underset{(c)}{- \vec{\mathscr{r}} \times (\vec{a} \times \nabla t_r)} + \underset{(d)}{\vec{v} \times (\vec{v} \times \nabla t_r)}$$

$$= \vec{v} + \cancel{\vec{a}(\vec{\mathscr{r}}\cdot\nabla t_r)} - \underbrace{\vec{\mathscr{r}} \times (\vec{a} \times \nabla t_r)}_{\substack{\cancel{\vec{a}(\vec{\mathscr{r}}\cdot\nabla t_r)} - \\ \nabla t_r (\vec{\mathscr{r}}\cdot\vec{a}) \\ \text{front Griffiths (2)}}} - \cancel{\vec{v}(\vec{v}\cdot\nabla t_r)} + \underbrace{\vec{v} \times (\vec{v} \times \nabla t_r)}_{\substack{\cancel{\vec{v}(\vec{v}\cdot\nabla t_r)} \\ - \nabla t_r (\vec{v}\cdot\vec{v}) \to v^2 \\ \text{front Griffiths (2)}}}$$

$$= \vec{v} + (\nabla t_r)(\vec{\mathscr{r}}\cdot\vec{a} - v^2) \qquad (9.100)$$

All together:

$$\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{1}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^2}\left[\vec{v} + (c^2 - v^2 + (\vec{\mathscr{r}}\cdot\vec{a}))\nabla t_r\right] \qquad (9.101)$$

From $\nabla(\mathscr{r}c)$

So the next step is to get ∇t_r

Note that $\underbrace{|\vec{\mathscr{r}}|}_{|\vec{r}-\vec{w}|} = c\,(t-t_r)$

Use (9.92) first:

$$-c\nabla t_r = \nabla \mathscr{r} = \nabla\sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}} = \frac{1}{2\sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}}}\nabla(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}) =$$

$$\underset{\uparrow}{=} \frac{1}{2\mathscr{r}}\cdot\left[2(\vec{\mathscr{r}}\cdot\nabla)\vec{\mathscr{r}} + 2(\vec{\mathscr{r}}\times(\nabla\times\vec{\mathscr{r}}))\right] = \frac{1}{\mathscr{r}}\Big[\underbrace{(\vec{\mathscr{r}}\cdot\nabla)\vec{\mathscr{r}}}_{(9.95)} + \underbrace{\vec{\mathscr{r}}\times(\nabla\times\vec{\mathscr{r}})}_{\vec{\mathscr{r}}\times(-\nabla\times\vec{w}) = \vec{\mathscr{r}}\times(\vec{v}\times\nabla t_r)}\Big]$$

Using (9.93) but with $\vec{v}\to\vec{\mathscr{r}}$

$\uparrow$ (9.98) (9.99)

(9.95)

$$(\vec{\mathscr{r}}\cdot\nabla)\vec{\mathscr{r}} = \underbrace{(\vec{\mathscr{r}}\cdot\nabla)\vec{r}}_{\vec{\mathscr{r}}\text{ as in (9.96)}} - \underbrace{(\vec{\mathscr{r}}\cdot\nabla)\vec{w}}_{\vec{v}(\vec{\mathscr{r}}\cdot\nabla t_r)\text{ as in (9.94)}}$$

$$-c\nabla t_r = \frac{1}{\mathscr{r}}\Big[\vec{\mathscr{r}} - \cancel{\vec{v}(\vec{\mathscr{r}}\cdot\nabla t_r)} + \underbrace{\vec{\mathscr{r}}\times(\vec{v}\times\nabla t_r)}_{\cancel{\vec{v}(\vec{\mathscr{r}}\cdot\nabla t_r)} - (\nabla t_r)(\vec{\mathscr{r}}\cdot\vec{v})}\Big] =$$

Griffiths front cover (1)

$$= \frac{1}{\mathscr{r}}\left(\vec{\mathscr{r}} - (\vec{\mathscr{r}}\cdot\vec{v})\nabla t_r\right)$$

Then: $$\boxed{\nabla t_r = \frac{\frac{\vec{\mathscr{r}}}{\mathscr{r}}}{-c + \frac{\vec{\mathscr{r}}\cdot\vec{v}}{\mathscr{r}}} = \frac{-\vec{\mathscr{r}}}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}}} \qquad (9.103)$$

Finally:

$$\boxed{\nabla V = \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})^3}\left[\vec{v}(\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v})-\left(c^2-v^2+(\vec{\mathscr{r}}\cdot\vec{a})\right)\vec{\mathscr{r}}\right]} \qquad (9.104)$$

when $\vec{v}=0,\ \vec{a}=0$, then:

$$\nabla V = \frac{qc}{4\pi\epsilon_0}\cdot\frac{1}{\mathscr{r}^3c^3}\left(-c^2\vec{\mathscr{r}}\right) = \frac{-q}{4\pi\epsilon_0}\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}$$

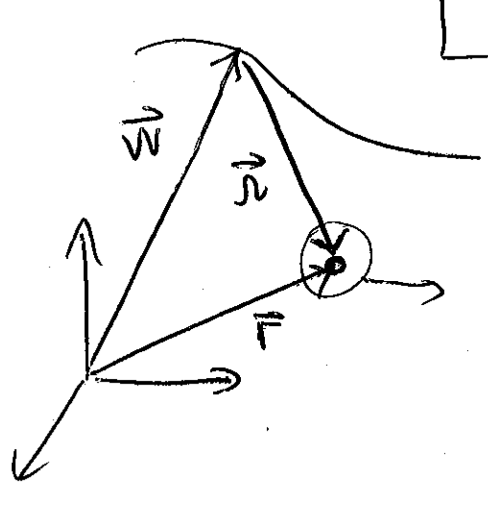
as it has to be for a static charge.

Following similar procedures we can calculate $\frac{\partial\vec{A}}{\partial t}$, and from $\vec{E}=-\nabla V-\frac{\partial\vec{A}}{\partial t}$ and $\vec{B}=\nabla\times\vec{A}$ we can get $\vec{E}$ and $\vec{B}$. The formulas look ugly, but they can be simplified to show that

$$\boxed{\vec{B}=\frac{1}{c}\left(\hat{\mathscr{r}}\times\vec{E}\right)} \qquad (9.108)$$

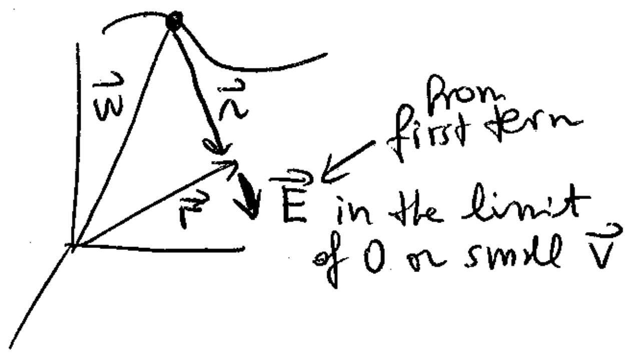
So $\vec{B}$ is always $\perp$ to $\hat{\mathscr{r}}$ and $\vec{E}$

unit vector from retarded point.



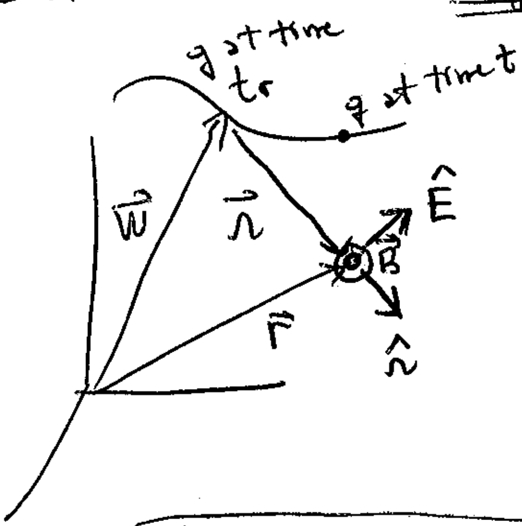
Note: in general $\vec{E}$ is not $\perp$ to $\hat{\mathscr{r}}$, see next page.

With regards to $\vec{E}$, it has 2 terms in (9.107). The <u>first term</u> is the "generalized Coulomb field" and if $\vec{v}=0$, then it becomes the electrostatic result $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$. So in this case $\vec{E}$ points along $\hat{\mathscr{r}}$, not $\perp$.



The <u>second term</u> cancels if $\vec{a}=0$, and it has one more power of $|\vec{\mathscr{r}}|$. Thus it is the one that matters for radiation (it is the "radiation field" or "acceleration field")

Since it is proportional to $\vec{\mathscr{r}} \times (\text{vector})$, then $\vec{E}$ of acceleration field is $\perp$ to $\hat{\mathscr{r}}$. Thus, the contribution at <u>large distances</u> is such that



i.e. $\vec{E}$ and $\vec{B}$ mutually $\perp$, and $\perp$ to $\hat{\mathscr{r}}$.

vector from <u>retarded time</u>, not from actual location of charge

So it behaves very much as any radiation field.

A charge must accelerate in order to radiate.