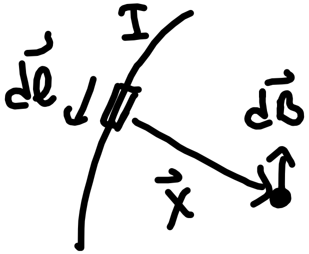
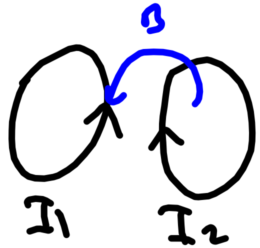
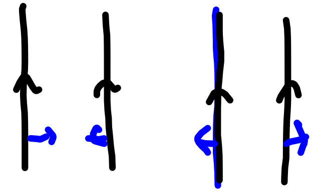
Ch. 5

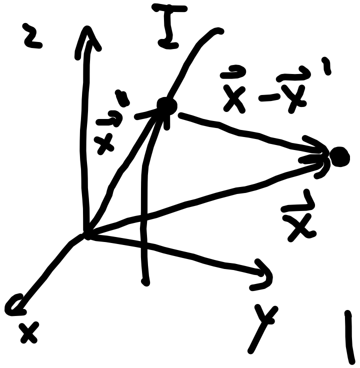
$$\nabla \cdot \vec{J} + \cancel{\frac{\partial \rho}{\partial t}} = 0$$



$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{x}}{|\vec{x}|^3}$$

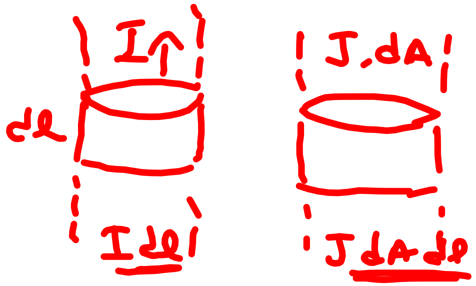






$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

obs. point ($\vec{x}$) source ($\vec{x}'$)



analog of $q \to \rho$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{x}')\times(\vec{x}-\vec{x}')}{\underbrace{|\vec{x}-\vec{x}'|^3}_{-\nabla_x\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)}}\, d^3x'$$

out integral

HW

$$= \frac{\mu_0}{4\pi}\left[\nabla_x \times \int \frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}\right] \leftarrow \nabla_x \times \left(\frac{\mu_0}{4\pi}\int \frac{\vec{J}\,d^3x'}{|\vec{x}-\vec{x}'|}\right)$$

$$\nabla\cdot\vec{B} = \nabla\cdot\left[\nabla\times\int\cdots\right] = 0$$

$$\nabla\times\vec{B} = \mu_0\vec{J} \;;\quad \nabla\times(\nabla\times\vec{a}) = \nabla(\nabla\cdot\vec{a}) - \nabla^2\vec{a}$$

HW

$$\nabla\cdot J = 0$$

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')$$

$$\boxed{\nabla\cdot\vec{B} = 0,\ \nabla\times\vec{B} = \mu_0\vec{J}}\quad \text{analog}\quad \boxed{\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0};\ \nabla\times\vec{E} = 0}$$

5.4 Vector Potential

$$\nabla\cdot\vec{B}=0\ , \quad \vec{B}=\nabla\times\vec{A}$$

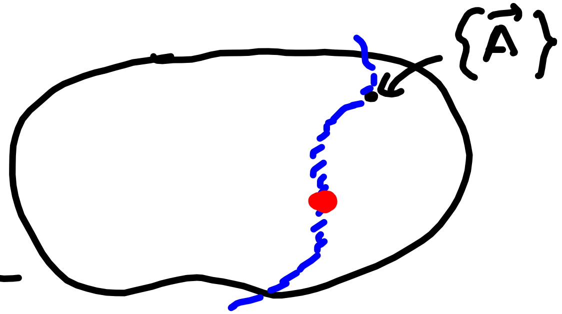
$$\nabla\cdot\vec{B}=\nabla\cdot(\nabla\times\vec{A})=0$$

$$\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$$

analog of $\phi=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$

$$\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\int\cdots + \nabla\psi(\vec{x})$$

ψ: scalar function



$$\nabla\times\vec{B}=\mu_0\vec{J}=\nabla\times(\nabla\times\vec{A})$$

$$=\nabla(\underbrace{\nabla\cdot\vec{A}}_{=0})-\nabla^2\vec{A}$$

$=0$ Coulomb gauge

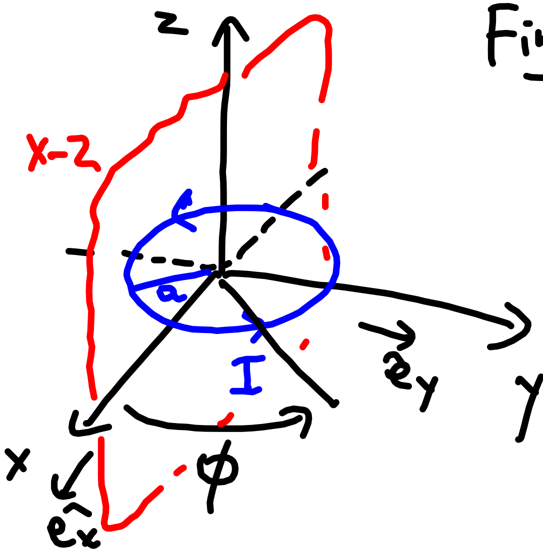
$$\nabla^2\vec{A}=-\mu_0\vec{J}$$

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

analog of $\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$

Derived under the Coulomb gauge $\nabla \cdot \vec{A} = 0$

5.5 Example



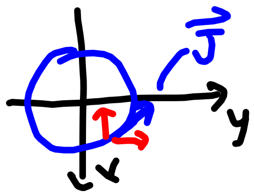
Find $\vec{B}$

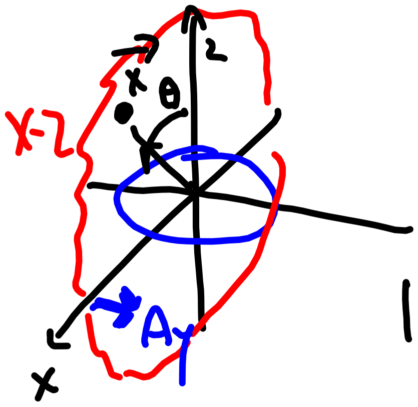
$$J_\phi = \frac{I}{a}\,\delta(\cos\theta')\,\delta(r'-a)$$

$$\int dx\, \delta(x) = 1$$

$L \qquad \frac{1}{L}$

$$\vec{J} = \cos\phi\, J_\phi\, \hat{e}_y - \sin\phi\, J_\phi\, \hat{e}_x$$



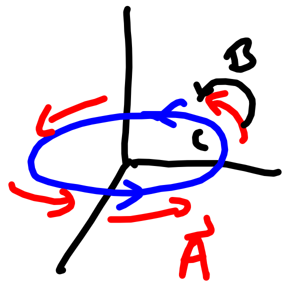


$$\vec{x} = (r\sin\theta,\ 0,\ r\cos\theta)$$

$$\vec{x}' = (a\cos\phi',\ a\sin\phi',\ 0)$$

$$|\vec{x}-\vec{x}'| = \sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$$



$$A_x = 0$$

$$A_y \neq 0 \leftarrow A_\phi = \frac{\mu_0}{4\pi} I a \int_0^{2\pi} \frac{\cos\phi'\,d\phi'}{\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}}$$

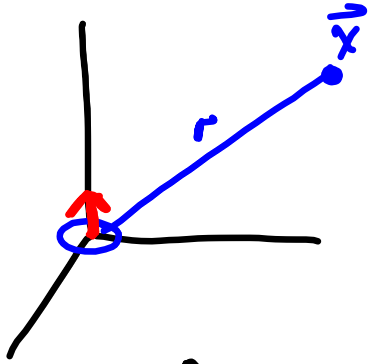
$$A_z = 0$$

$$A_\theta = 0$$

$$A_r = 0$$

$$\vec{B} = \nabla\times\vec{A} = \hat{e}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) + \cdots$$

$r \gg a$



$$\frac{1}{\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}} \approx \frac{1}{r}\left(1+\frac{a}{r}\sin\theta\cos\phi'\right)$$

$$A_\phi \approx \frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\cos\phi'\,d\phi'\,\frac{1}{r}\left(\cancel{1}+\frac{a}{r}\sin\theta\cos\phi'\right)$$

$$\int_0^{2\pi}\cos^2\phi'\,d\phi' = \pi \qquad = \boxed{\frac{\mu_0}{4\pi}\,\frac{I\pi a^2\sin\theta}{r^2}}$$

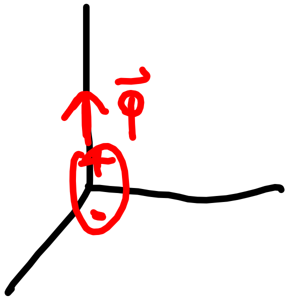
$$\boxed{\begin{aligned} B_r &= \frac{\mu_0}{2\pi}\,\frac{(I a^2\pi)}{r^3}\cos\theta \\ B_\theta &= \frac{\mu_0}{4\pi}\,\frac{\overbrace{(I a^2\pi)}^{m}}{r^3}\sin\theta \end{aligned}}$$

m = magnetic dipole moment

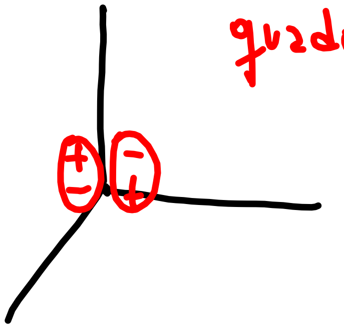
Like a dipole

$\vec{E}$

$$E_r^{dipole} = \frac{1}{2\pi\epsilon_0}\frac{p\cos\theta}{r^3}$$; Ch4

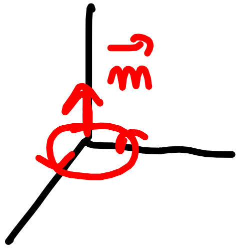


electric quadrupole



$\vec{B}$

$$B_r^{dipole} = \frac{\mu_0}{2\pi}\frac{m\cos\theta}{r^3}$$



magnetic quadrupole