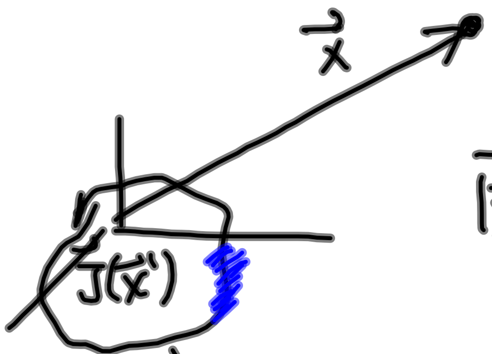
## 5.6 Mag. fields of a localized current distribution




$\nabla \cdot \vec{J} = 0$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|\left|\vec{n}-\frac{\vec{x}'}{|\vec{x}|}\right|} = \frac{1}{|\vec{x}|\sqrt{\left(\vec{n}-\frac{\vec{x}'}{|\vec{x}|}\right)\cdot\left(\vec{n}-\frac{\vec{x}'}{|\vec{x}|}\right)}} \approx$$

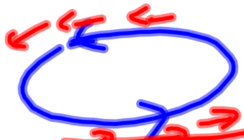
$\vec{x} = \vec{n}|\vec{x}|$

$$\approx \frac{1}{|\vec{x}|}\frac{1}{\sqrt{1-2\frac{\vec{n}\cdot\vec{x}'}{|\vec{x}|}}} \approx \frac{1}{|\vec{x}|}\left(1+\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2}\right)$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|} \approx \frac{\mu_0}{4\pi}\int\vec{J}(\vec{x}')\left[\frac{1}{|\vec{x}|}+\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3}\right]d^3x'$$

$$\int \vec{J}(\vec{x}')\, d^3x' = 0 \qquad \nabla \cdot \vec{J} = 0$$

No monopoles



$\vec{m}$ magnetic moment

$$\vec{A}(\vec{x}) \approx \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}')\, \frac{(\vec{x}\cdot\vec{x}')}{|\vec{x}|^3}\, d^3x' = \frac{\mu_0}{4\pi}\, \frac{\left[\frac{1}{2}\int (\vec{x}' \times \vec{J})\, d^3x'\right] \times \vec{x}}{|\vec{x}|^3} =$$

$$\frac{\vec{x}}{|\vec{x}|} = \vec{n}$$

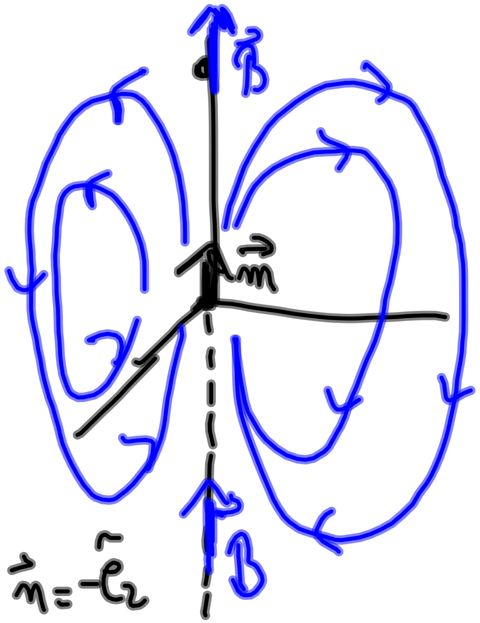
$$= \frac{\mu_0}{4\pi}\left[\frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}\right]$$

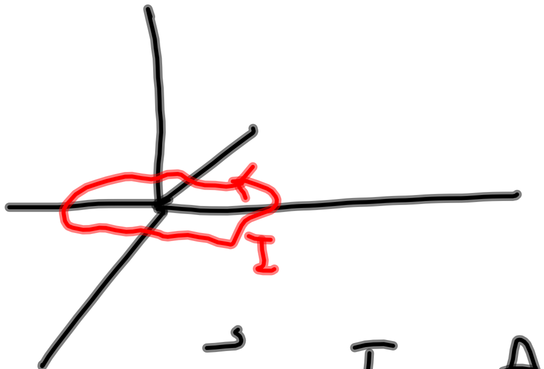
localized sources $\nabla \cdot \vec{J} = 0$

$$\boxed{\vec{B} = \nabla \times \vec{A} = \nabla \times \left[\frac{\mu_0}{4\pi}\, \frac{(\vec{m} \times \vec{x})}{|\vec{x}|^3}\right] = \frac{\mu_0}{4\pi}\, \frac{\left[3\vec{n}(\vec{n}\cdot\vec{m}) - \vec{m}\right]}{|\vec{x}|^3}}$$

$\vec{m} = m\hat{e}_z \quad , \quad \vec{n} = \hat{e}_z$

$3\,\vec{n}(\vec{n}\cdot\vec{m}) - \vec{m}$




$\vec{m} = I \times \text{Area}_{\text{loop}}$

$\vec{m} = I(\pi a^2)$

# 5.8 Macroscopic equations ; Boundary conditions

$$\nabla \cdot \vec{B} = 0$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \left[ \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\vec{M}(\vec{x}') \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} \right] d^3x'$$


Chapter 4

$$\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = \nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\overbrace{\left[\vec{J}(\vec{x}\,') + \nabla' \times \vec{M}(\vec{x}\,')\right]}^{\vec{J}_{eff}(\vec{x}\,')}}{|\vec{x}-\vec{x}\,'|} d^3x'$$

$$\nabla \times \vec{B} = \mu_0 \left[\vec{J} + \nabla \times \vec{M}\right]$$

$$\vec{H} \underset{\text{def}}{=} \frac{\vec{B}}{\mu_0} - \vec{M}$$

(analog of $\vec{D}$)

$$\nabla \times \vec{H} = \nabla \times \frac{\vec{B}}{\mu_0} - \nabla \times \vec{M} = \vec{J} \, ,$$

$$\boxed{\nabla \times \vec{H} = \vec{J}\,,}$$

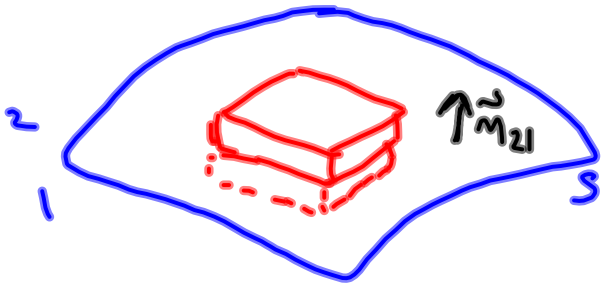
$$\boxed{\nabla \cdot \vec{B} = 0}$$

$\vec{H} = \vec{H}(\vec{B})$ ?

linear and isotropic $\vec{B} = \mu \vec{H}$

magnetic permeability

## Boundary conditions




$$\int \nabla \cdot \vec{B}\, d^3x' = 0 = \oint_S \vec{B} \cdot \vec{n}\, da$$

$$\boxed{(\vec{B}_2 - \vec{B}_1) \cdot \vec{n}_{21} = 0}$$

S

Stokes theorem

$$\int_S (\nabla\times\vec{C})\cdot d\vec{S} = \int_{\text{line}} \vec{C}\cdot d\vec{\ell}$$

$$\vec{n}_{21}\times(\vec{H}_2-\vec{H}_1) = \vec{K}$$

surface current

(analog of $\sigma$)

$$(\vec{B}_2-\vec{B}_1)\cdot\vec{n}_{21} = 0$$

$$\nabla\times\vec{H} = \vec{J}, \quad \nabla\cdot\vec{B} = 0, \quad \vec{B} = \mu\vec{H}$$

# 5.9 How do we solve these problems?

(A) $\nabla\cdot\vec{B}=0 \longrightarrow \vec{B}=\nabla\times\vec{A}$

linear $\vec{B}=\mu\vec{H}$

$$\nabla\times\vec{H}=\vec{J} \longrightarrow \nabla\times\frac{\vec{B}}{\mu}=\underbrace{\nabla\times\left(\frac{1}{\mu}\nabla\times\vec{A}\right)}=\vec{J}$$

$$\nabla\times(\nabla\times\vec{A})=\nabla(\underbrace{\nabla\cdot\vec{A}}_{=0})-\nabla^2\vec{A}$$

Coul. gauge

$$\boxed{\nabla^2\vec{A}=-\mu\vec{J}}$$

linear medium

Coulomb gauge

B $\quad \vec{J} = 0\ , \quad \nabla \times \vec{H} = 0$

$$\vec{H} = -\nabla \Phi_M$$

magnetic scalar potential

$$\nabla \cdot \vec{B} = \mu \nabla \cdot \vec{H} = \mu\, \nabla \cdot \left(-\nabla \Phi_M\right) = 0$$

linear medium

$$\boxed{\nabla^2 \Phi_M = 0}$$