5.9 How to solve problems?

(A) $\nabla \cdot \vec{B} = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}$

Linear $\vec{B} = \mu \vec{H}$. If $\vec{J} \neq 0$, then:

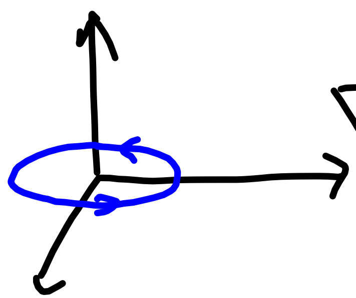
$$\nabla \times \vec{H} = \vec{J} \longrightarrow \nabla \times \left(\frac{\vec{B}}{\mu}\right) = \frac{1}{\mu} \nabla \times (\nabla \times \vec{A}) = \frac{1}{\mu}\Big[\nabla(\underbrace{\nabla \cdot \vec{A}}_{=0}) - \nabla^2 \vec{A}\Big] = \vec{J}$$

Coul. gauge

$$\boxed{\nabla^2 \vec{A} = -\mu \vec{J}}$$

Linear material; Coul. gauge.

Example



$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

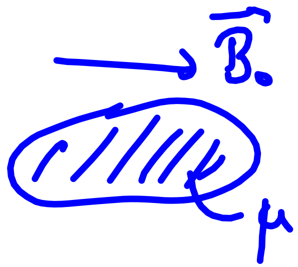
(B) If $\vec{J}=0 \rightarrow \nabla\times\vec{H}=0 \rightarrow \vec{H}=-\nabla\Phi_M$ "magnetic scalar potential"

$$\nabla\cdot\vec{B}=0 \overset{\text{if linear}}{=} \mu\,\nabla\cdot\vec{H} = -\mu\nabla^2\Phi_M$$

$$\boxed{\nabla^2\Phi_M=0}$$

Laplace Eq. as in Electrostatics

Example



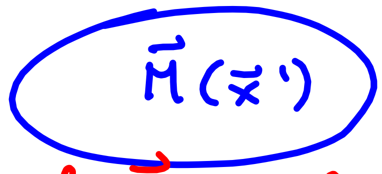
(C) Hard ferromagnet. $\vec{M}\neq 0$, fixed

Assume $\vec{J}=0 \rightarrow \vec{H}=-\nabla\Phi_M$

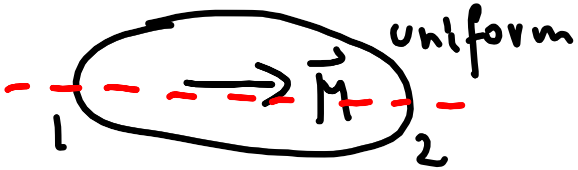
$$\nabla\cdot\vec{B}=0=\nabla\cdot\left[\mu_0(\vec{H}+\vec{M})\right]$$

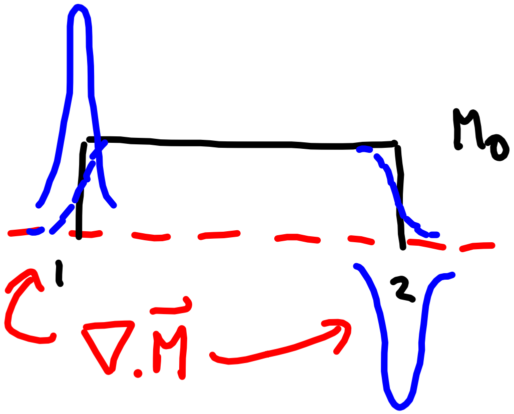
$$\boxed{\nabla^2\Phi_M=\nabla\cdot\vec{M}}$$

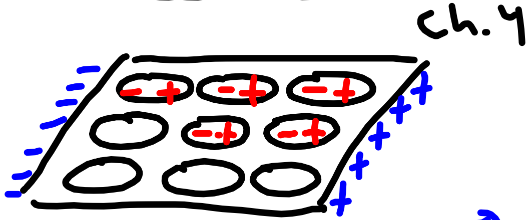
Example

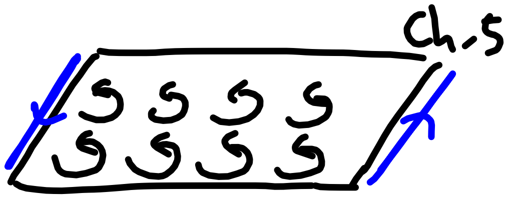


If $\vec{M}$ uniform then sources of ... are at the bounda...







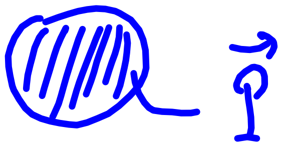


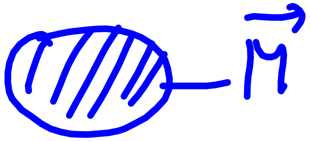
$\vec{n}'\cdot\vec{P}$

$$\Phi_E = \frac{1}{4\pi}\oint_S \frac{\sigma_E \, ds'}{|\vec{x}-\vec{x}'|}$$

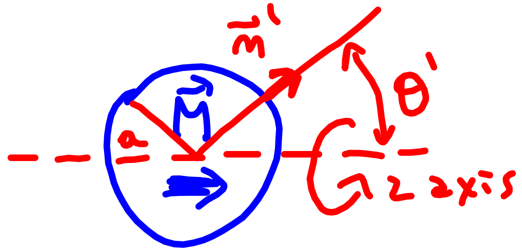
$\vec{n}'\cdot\vec{M}$

$$\Phi_M = \frac{1}{4\pi}\oint_S \frac{\sigma_M \, ds'}{|\vec{x}-\vec{x}'|}$$





5.10 Example of a uniformly magnetized sphere



$\vec{B}, \vec{H}$?

Sources of $\vec{H}$ are at the boundary

Since $\vec{J}=0 \rightarrow \Phi_M = \frac{1}{4\pi}\oint_S \frac{\vec{n}'\cdot\vec{M}\, ds'}{|\vec{x}-\vec{x}'|} = \frac{1}{4\pi}\int_S \frac{M_0 \cos\theta'\, a^2 d\Omega'}{|\vec{x}-\vec{x}'|}$

$\uparrow$ obs. point

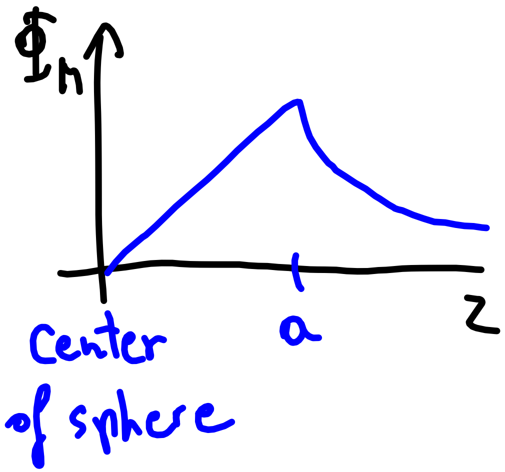
$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l,m} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} \underbrace{Y^*_{lm}(\theta',\phi')}_{\text{related to } \vec{x}'} \underbrace{Y_{lm}(\theta,\phi)}_{\text{related to } \vec{x}}$$

$r_<$ = smallest of r and a

$r_>$ = largest of r and a

$$\cos\theta' = \sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta', \phi') \;; \quad \int d\Omega'\, Y^*_{lm}(\theta', \phi')\, Y_{10}(\theta', \phi') = \delta_{l1}\,\delta_{m0}$$

$$\Phi_M = \begin{cases} \frac{M_0}{3} r\cos\theta = \frac{M_0}{3} z & \text{Inside } (r_< = r,\ r_> = a) \\ \frac{M_0}{3}\left(\frac{a}{r}\right)^3 z & \text{Outside } (r_< = a,\ r_> = r) \end{cases}$$



Φ_M is continuous

Inside: $\Phi_M = \frac{M_0}{3} z \; ; \; \vec{H} = -\nabla \Phi_M = -\frac{M_0}{3}\hat{e}_z$

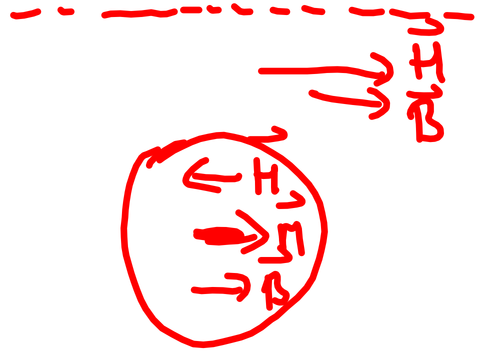
$$\vec{B} \underset{\text{def.}}{=} \mu_0(\vec{H}+\vec{M}) = \frac{2}{3}\mu_0 M_0 \hat{e}_z$$

Clearly $\vec{B} \neq \mu \vec{H}$

Outside: $\vec{M} = 0 \; , \; \Phi_M = \frac{M_0}{3}\left(\frac{a}{r}\right)^3 z$

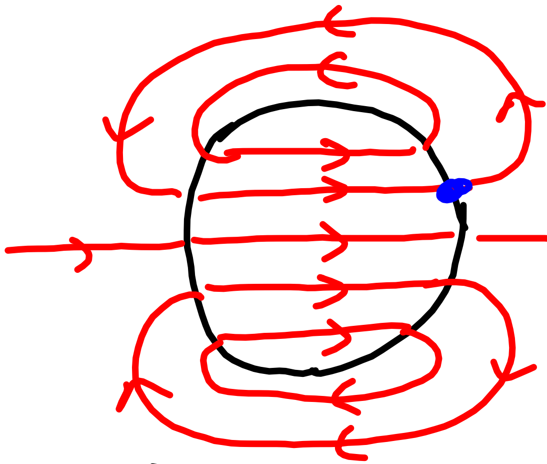
$$\vec{H} = -\nabla \Phi_M = \frac{2}{3} M_0 a^3 \frac{1}{z^3}\hat{e}_z$$

↑ along the z axis



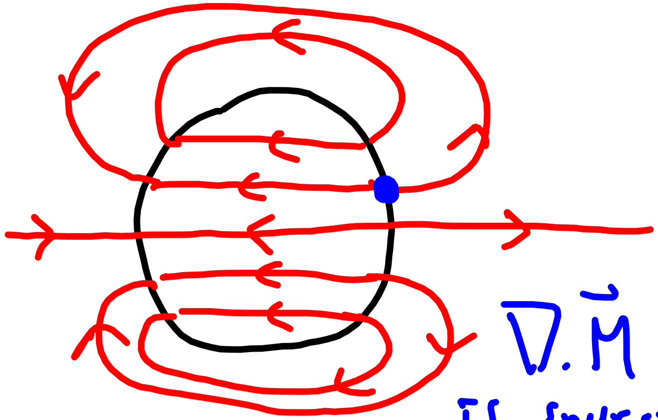
$$\vec{B} = \mu_0 \vec{H}$$

$\vec{B}$ continuous at surface because $\nabla \cdot \vec{B} = 0$ | $\vec{H}$ discontinuous at surface because of $\nabla \cdot \vec{M}$



$\nabla \cdot \vec{B} = 0$

$\vec{B}$ continuous

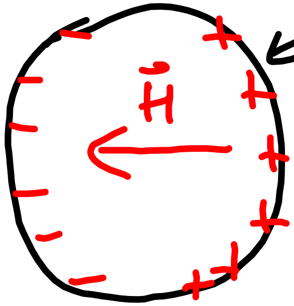


$\nabla \cdot \vec{M}$ is source of $\vec{H}$

$\vec{H}$ discontinuous

$$\nabla^2 \Phi_M = \nabla \cdot \vec{M}$$

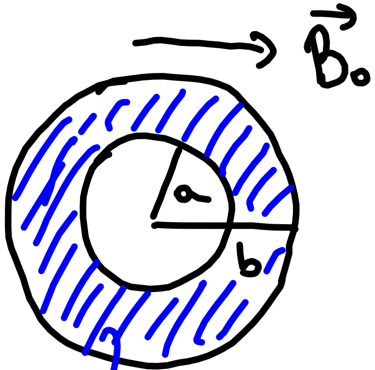
"like $-\frac{\rho}{\epsilon_0}$ in Poisson Eq."



$\nabla \cdot \vec{M} < 0$ is like $\rho > 0$

5.12 Magnetic Shielding

HW



linear

$\vec{B} = \mu \vec{H}$

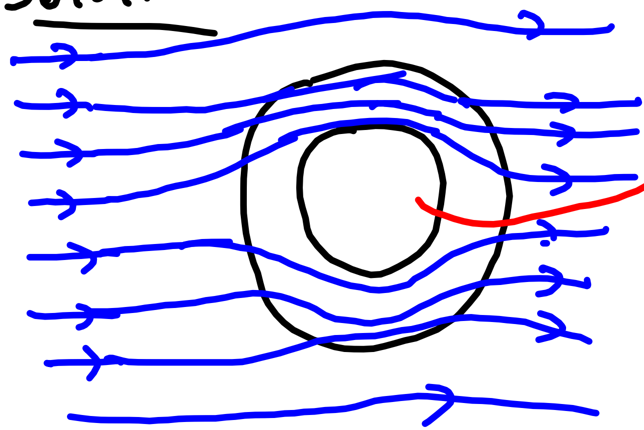
Find $\vec{H}$ and $\vec{B}$ everywhere

Path Ⓑ. Use Φ_M since $\vec{J} = 0$

+ boundary conditions for $\vec{H}$ and $\vec{B}$

Same as electrostatic problem

Solution



$|B_{inside}| \sim \frac{\mu_0}{\mu}$

$\mu \sim 10^5 \mu_0$