6.1 Maxwell equations

So far :

$$\nabla\cdot\vec{B}=0 \qquad \nabla\times\vec{H}=\vec{J} \quad \leftarrow \text{incomplete}$$

$$\nabla\cdot\vec{D}=\rho \qquad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

For dynamical systems $\rightarrow$ $\underset{\text{ch5}}{\nabla\cdot\vec{J}=0}$ $\longrightarrow$ $\nabla\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$ Continuity Eq.

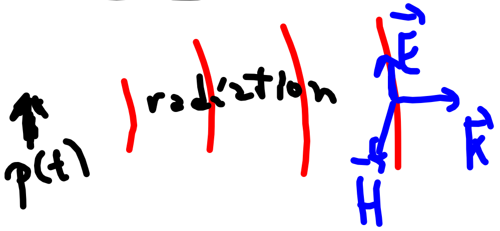
$$\nabla\cdot\vec{J}+\frac{\partial}{\partial t}(\nabla\cdot\vec{D})=0$$

$$\nabla\cdot\underbrace{\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)}_{\vec{J}_{eff}}=0$$

$$\boxed{\begin{array}{ll}\nabla\cdot\vec{B}=0 & \nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t} \\ \nabla\cdot\vec{D}=\rho & \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\end{array}}$$

$\frac{\partial\vec{D}}{\partial t}$ source of $\vec{H}$

$\frac{\partial\vec{B}}{\partial t}$ source of $\vec{E}$



6.2 Vector and Scalar Potentials

$$\nabla\cdot\vec{B}=0 \longrightarrow \boxed{\vec{B}=\nabla\times\vec{A}}$$

$$\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \longrightarrow \nabla\times\vec{E}+\frac{\partial}{\partial t}(\nabla\times\vec{A})=0\ ,\ \nabla\times\underbrace{\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)}_{-\nabla\phi}=0$$

$$\boxed{\vec{E}=-\nabla\phi-\frac{\partial\vec{A}}{\partial t}}$$

$$\nabla\cdot\vec{D}=\rho\ ;\ \underset{\text{vacuum}}{\vec{D}=\epsilon_0\vec{E}} \qquad \epsilon_0\nabla\cdot\left(-\nabla\phi-\frac{\partial\vec{A}}{\partial t}\right)=\rho$$

$$\text{I} \qquad \boxed{\nabla^2\phi+\frac{\partial}{\partial t}(\nabla\cdot\vec{A})=-\frac{\rho}{\epsilon_0}}$$

Replaces the Poisson Eq.

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$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

vacuum

$$\frac{1}{\mu_0}(\nabla \times \vec{B}) = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\frac{1}{\mu_0}\left[\nabla \times (\nabla \times \vec{A})\right] = \vec{J} + \epsilon_0 \frac{\partial}{\partial t}\left(-\nabla \phi - \frac{\partial \vec{A}}{\partial t}\right)$$

$$\frac{1}{\mu_0}\left[\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}\right] = \vec{J} - \nabla\left(\epsilon_0 \frac{\partial \phi}{\partial t}\right) - \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\boxed{\nabla^2 \vec{A} - \underbrace{\mu_0 \epsilon_0}_{\frac{1}{c^2}} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla\left(\nabla \cdot \vec{A} + \underbrace{\mu_0 \epsilon_0}_{1/c^2} \frac{\partial \phi}{\partial t}\right) = -\mu_0 \vec{J}} \quad \text{II}$$

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gauge invariance

$$\vec{B} = \nabla \times \vec{A} \longrightarrow \left.\vec{A}' = \vec{A} + \nabla\Lambda\right\} \text{same } \vec{B}$$

scalar

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \quad \left.\begin{array}{l} \vec{A}' = \vec{A} + \nabla\Lambda \\ \phi' = \phi - \dfrac{\partial \Lambda}{\partial t} \end{array}\right\} \text{same } \vec{E} \qquad \Lambda = \Lambda(\vec{x}, t)$$

$$\boxed{\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0}$$

Lorentz gauge

II $\longrightarrow$ $$\boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}}$$

I $\longrightarrow$ $$\boxed{\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}}$$

$$\left.\begin{array}{l} \vec{x},\ ct \\ \vec{A},\ \phi \end{array}\right\} \text{formally ``similar''}$$

Coulomb gauge

$$\nabla \cdot \vec{A} = 0$$

(I) $$\nabla^2 \phi(\vec{x},t) = -\frac{\rho}{\epsilon_0}(\vec{x},t)$$

$$\phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}',t)\, d^3x'}{|\vec{x}-\vec{x}'|}$$ Instantaneous!

$$\vec{E} = -\nabla\phi - \underbrace{\frac{\partial \vec{A}}{\partial t}}_{\text{not instantaneous}}$$

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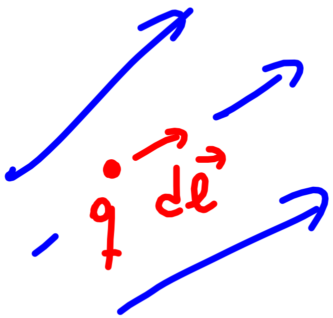
6.7 Conservation of energy



W = Work by the fields

$$\underbrace{\frac{dW}{dt}}_{\text{Power}} = \vec{F}\cdot\frac{d\vec{l}}{dt} = q\vec{E}\cdot\overbrace{\frac{d\vec{l}}{dt}}^{\vec{v}} = q\vec{E}\cdot\vec{v}$$

$$\sum_i q_i \vec{E}_i\cdot\vec{v}_i$$

current

$$\int_V \vec{J}(\vec{x},t)\cdot\vec{E}(\vec{x},t)\,d^3\vec{x}$$

$$\boxed{\frac{dW}{dt} = \int_V \vec{J}\cdot\vec{E}\,d^3x}$$

$$\equiv \frac{dE_{particles}}{dt}$$

$$\int \vec{J}\cdot\vec{E}\, d^3x = \int d^3x\, \vec{E}\cdot\Big(\underbrace{\nabla\times\vec{H} - \frac{\partial \vec{D}}{\partial t}}_{\text{Max Eq.}}\Big) =$$

$$\vec{E}\cdot(\nabla\times\vec{H}) = -\nabla\cdot(\vec{E}\times\vec{H}) + \vec{H}\cdot(\nabla\times\vec{E})$$

math

$$= \int d^3x \Big[-\nabla\cdot(\vec{E}\times\vec{H}) + \vec{H}\cdot\underbrace{(\nabla\times\vec{E})}_{-\frac{\partial \vec{B}}{\partial t}\ \text{Max. Eq.}} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\Big]$$

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} \overset{\text{linear}}{=} \epsilon\left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right) = \frac{\epsilon}{2}\frac{\partial(\vec{E}\cdot\vec{E})}{\partial t} = \frac{1}{2}\frac{\partial(\vec{E}\cdot\vec{D})}{\partial t}$$

$$\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{2}\frac{\partial(\vec{H}\cdot\vec{B})}{\partial t}$$

$$\boxed{\int_V \vec{J}\cdot\vec{E}\, d^3x = -\underbrace{\int_V d^3x\, \nabla\cdot(\vec{E}\times\vec{H})}_{\text{surface}} - \frac{\partial}{\partial t}\int_V d^3x\, \underbrace{\left(\frac{\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}}{2}\right)}_{\text{energy density "}u\text{"}}}$$

$$\frac{dE_{particles}}{dt} + \frac{dE_{fields}}{dt} + \text{energy lost at } S = 0$$

$$\vec{S} = \vec{E} \times \vec{H}$$

Poynting vector

Example