1.7 Poisson and Laplace Eqs.

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \times \vec{E} = 0$$

$\vec{E} = -\nabla \phi$

$$\boxed{\nabla^2 \Phi = -\frac{\rho(\vec{x})}{\epsilon_0}}$$ Poisson Eq.

$$\boxed{\nabla^2 \Phi = 0}$$ Laplace Eq.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

$$\nabla^2 \Phi = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}') \nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) d^3x' = -\frac{\rho(\vec{x})}{\epsilon_0}$$

$\nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = -4\pi \delta(\vec{x} - \vec{x}')$

Green's functions elementary introduction

$$\hat{L}u(x) = f(x)$$

given ? given

$$\hat{L}_x G(x,s) = \delta(x-s)$$

$\hat{L} = \frac{d}{dx^2}$

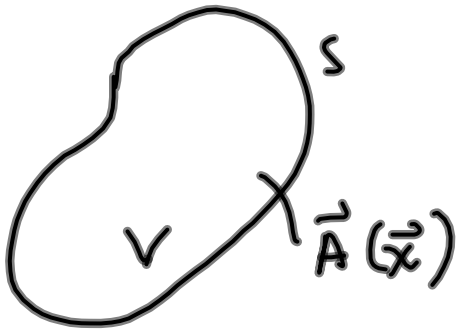
$$\int \hat{L} G(x,s) f(s)\, ds = \int \delta(x-s) f(s)\, ds = f(x) = \hat{L}u(x)$$

$$\hat{L}\int G(x,s) f(s)\, ds$$

$$\boxed{u(x) = \int G(x,s) f(s)\, ds}$$

1.8 Green's theorem

$$\int_V \nabla \cdot \vec{A}\, d^3x = \oint_S (\vec{A}\cdot\vec{n})\, da$$



$\vec{A} = \phi \nabla\psi$; ϕ, ψ are arbitrary

$$\nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$$

$$\vec{A}\cdot\vec{n} = \phi\nabla\psi\cdot\vec{n} = \phi\frac{\partial\psi}{\partial n}$$

$$\int_V [\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi]\, d^3x = \oint_S \phi\frac{\partial\psi}{\partial n}\, da$$

$\phi \to \psi$, $\psi \to \phi$ ——— difference

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, d^3x = \oint_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) da$$

Green's theorem

$\psi = \frac{1}{|\vec{x} - \vec{x}'|}$, $\phi = \Phi(\vec{x})$ $\quad \nabla^2$

Special case $\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$

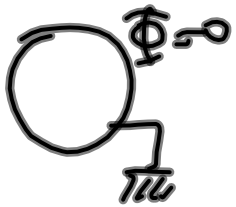
$$\int_V \left[\Phi(\vec{x}') \nabla'^2 \left(\frac{1}{|\vec{x} - \vec{x}'|}\right) - \frac{1}{|\vec{x} - \vec{x}'|}\left(\frac{-\rho(\vec{x}')}{\epsilon_0}\right)\right] d^3x' =$$

$\nabla'^2 \left(\frac{1}{|\vec{x} - \vec{x}'|}\right) = -4\pi\, \delta(\vec{x} - \vec{x}')$

$$= \oint_S \left[\Phi \frac{\partial}{\partial n'}\left[\frac{1}{|\vec{x} - \vec{x}'|}\right] - \frac{1}{|\vec{x} - \vec{x}'|} \frac{\partial \Phi}{\partial n'}\right] da'$$

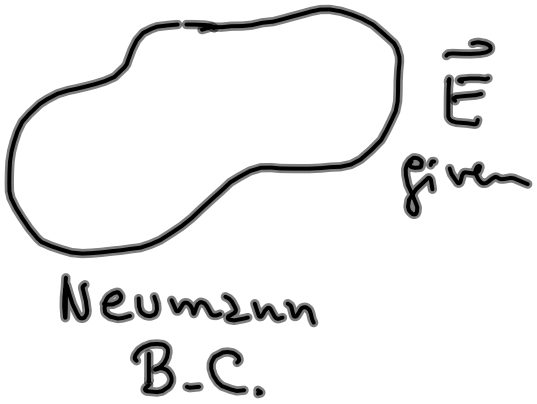
$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{|\vec{x}-\vec{x}'|}\frac{\partial\Phi(\vec{x}')}{\partial n'} - \Phi(\vec{x}')\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right] da'$$

given



if Φ and $\frac{\partial\Phi}{\partial n'}$ are known at S, then we solved the problem.

1.9 Boundary conditions



1.10 Green functions and value problems

$$\nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$$

$$G = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}')$$

$$\underbrace{\nabla^2 F(\vec{x},\vec{x}') = 0}_{\text{inside } V}$$

Back to Green's theorem, $\phi = \Phi$, $\psi = G(\vec{x},\vec{x}')$

$$\nabla^2 G = -4\pi\delta$$

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\,G(\vec{x},\vec{x}')\,d^3x'$$

$$+\frac{1}{4\pi}\oint_S \left[G(\vec{x},\vec{x}')\frac{\partial\Phi(\vec{x}')}{\partial n'} - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right]da'$$

$$\nabla^2 G = -4\pi\delta$$

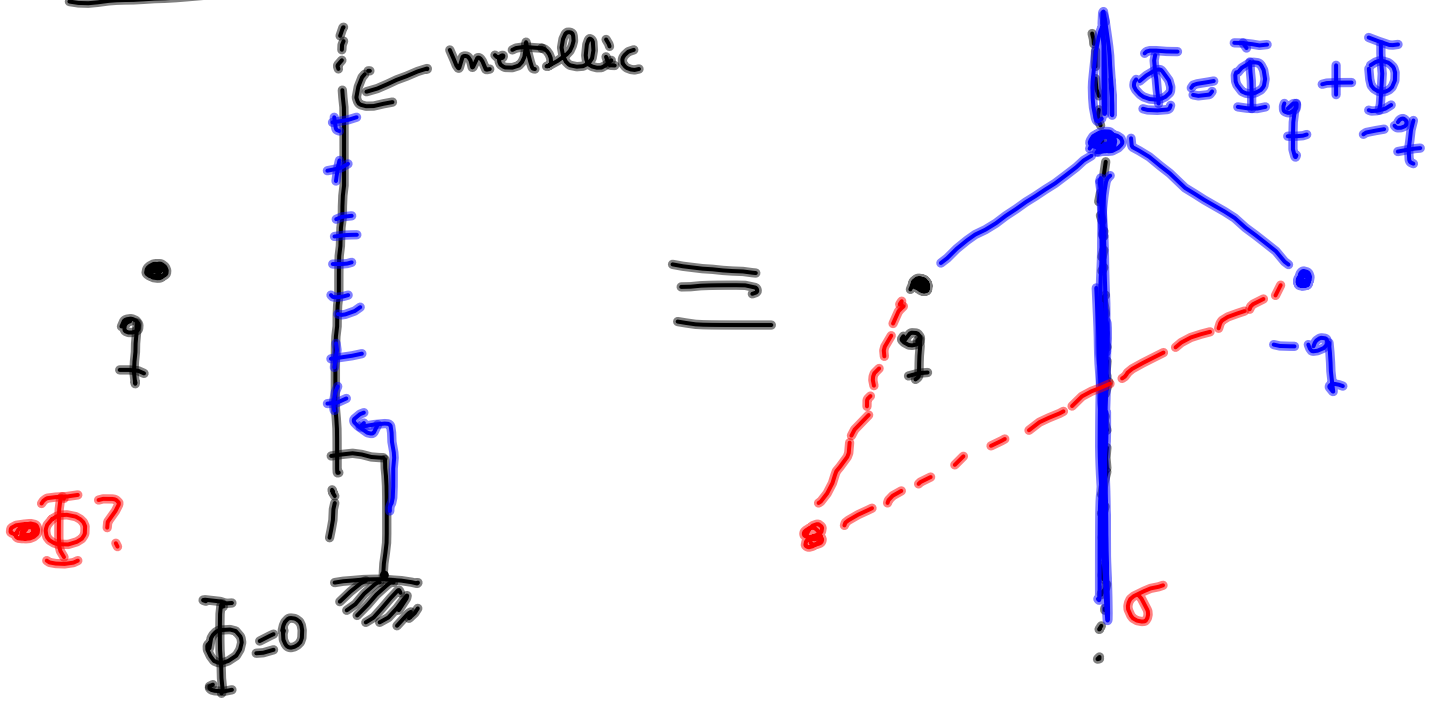
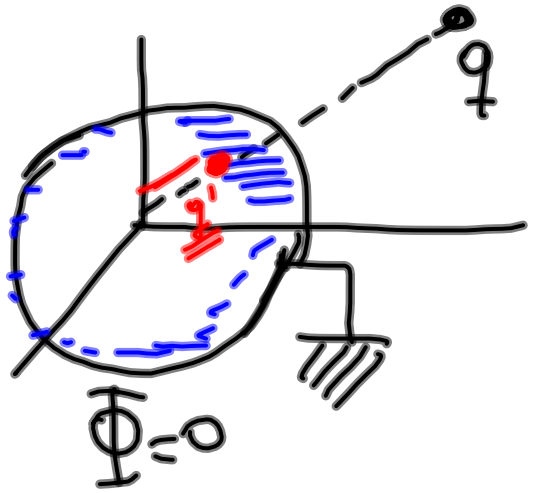
$$G(\vec{x},\vec{x}') = 0 \quad \text{if } \vec{x}' \text{ belongs to } S$$

$$G_D(\vec{x},\vec{x}')$$

↑ Dirichlet

Chapter 2

2.1 Method of images



2.2 Point charge in the presence of a conducting sphere