

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^n \sum_{\substack{j=1 \\ (i \neq j)}}^n \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$$

$$W = \frac{1}{8\pi\epsilon_0} \int \int \frac{\rho(\vec{x}) \rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x d^3x' = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) d^3x =$$

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

implies

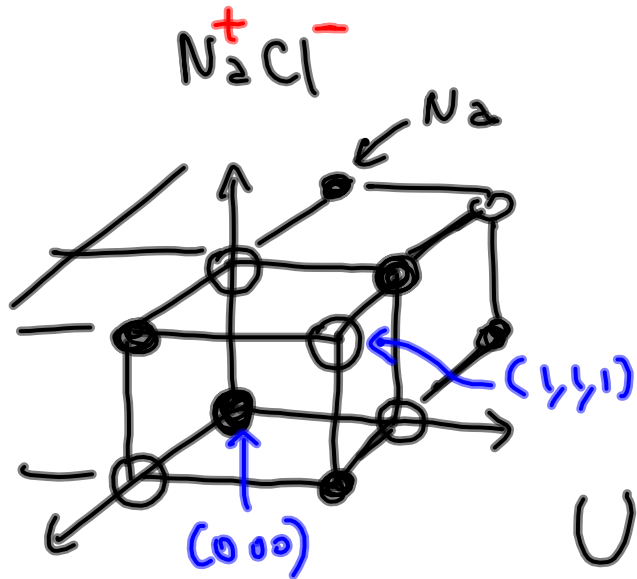
$$-\frac{\epsilon_0}{2} \int \phi \nabla^2 \phi d^3x = \frac{\epsilon_0}{2} \int |\nabla \phi|^2 d^3x = \frac{\epsilon_0}{2} \int \vec{E}^2 d^3x$$

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}') d^3x'}{|\vec{x} - \vec{x}'|}$$

$$\nabla \cdot (\phi \nabla \phi) = \underbrace{|\nabla \phi|^2}_{\text{localized charges}} + \phi \nabla^2 \phi$$

$$\int_V \nabla \cdot \vec{A} = \oint_S \vec{A} \cdot \vec{n} dS$$

Electrostatic energy of ionic crystals



$$\vec{m}_q = (m_x, m_y, m_z) a$$

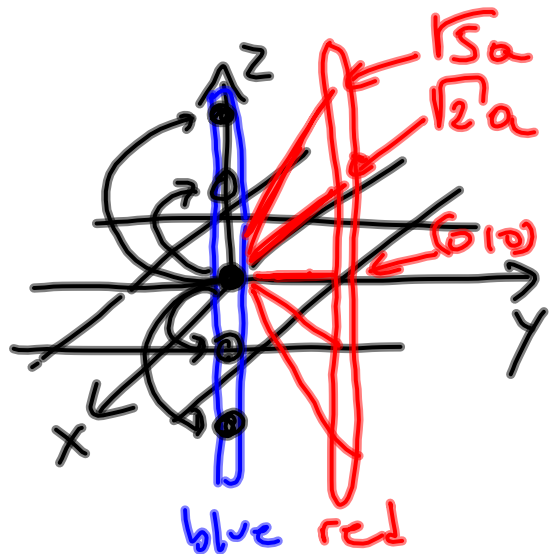
\uparrow
 $-2, -1, 0, 1, 2, \dots$

$\approx 2.8 \text{ \AA}$

$$U = \frac{e^2}{a} \sum_{\vec{m} \neq \vec{0}} (-1)^{m_x + m_y + m_z} \frac{1}{|\vec{m}|}$$

\uparrow
 $\sum \sum \sum_{m_x m_y m_z}$

$\int \frac{dx}{x} \sim \ln \Lambda$



blue red
 $U_0 + 4U_{10} + \dots$

$U_0 = \frac{ze^2}{a}$

$-\ln 2$

$$\left[-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots \right]$$

Diagram showing the series terms with arrows pointing to the U_0 term and the series terms. The terms are: -1 (pointing to $z > 0$), $+\frac{1}{2}$ (pointing to $z < 0$), $-\frac{1}{3}$ (pointing to $+$), $+\frac{1}{4}$ (pointing to $-$), and so on. The signs alternate as $+$, $-$, $+$, $-$, etc. for the terms in the series.

N_A

$$\frac{NU}{\text{moles}} = \frac{-Ne^4}{\text{moles } a} (1.7476)$$

"Madelung sums"

2.8 Orthogonal functions and expansions

$\{ (a, b) \quad U_n(x)$

orthogonal, square integrable

$$\int_a^b U_n^*(x) U_m(x) dx = \delta_{nm}$$

$$\int_a^b |U_n(x)|^2 dx = 1$$

$$f(x) = \sum_n a_n U_n(x)$$

$$\int_a^b U_m^*(x) f(x) dx = \sum_n a_n \underbrace{\int_a^b U_m^*(x) U_n(x) dx}_{\delta_{nm}} = a_m$$

$$\sum_n U_m^*(x') U_n(x) = \delta(x-x')$$

completeness relation

$$\sqrt{\frac{2}{a}} \sin\left(\frac{2\pi m x}{a}\right), \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi m x}{a}\right)$$


$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} \left[A_m \cos\left(\frac{2\pi m x}{a}\right) + B_m \sin\left(\frac{2\pi m x}{a}\right) \right]$$

$$A_m = \frac{2}{a} \int_{-a/2}^{+a/2} f(x) \cos\left(\frac{2\pi m x}{a}\right) dx$$

$$B_m \int_{-a/2}^{+a/2} f(x) \sin\left(\frac{2\pi m x}{a}\right) dx$$

$$a \rightarrow \infty, \quad \frac{2\pi m}{a} = k, \quad \sum_m \rightarrow \frac{a}{2\pi} \int_{-\infty}^{+\infty} dk$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{A(k)}_{\text{am}} e^{ikx} dk$$

sin, cos
 sin $\pm i\epsilon \rightarrow e^{ikx}, e^{-ikx}$

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(k-k')x} dx = \delta(k-k')$$

orthogonality

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-x')} dk = \delta(x-x')$$

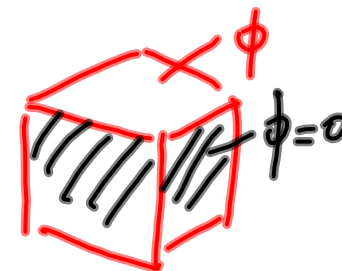
completeness

2.9 Separation of variables

Laplace eq. $\nabla^2 \phi = 0$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$\phi(x, y, z) = X(x) Y(y) Z(z)$ and then divide by XYZ



$$\underbrace{\frac{1}{X(x)} \frac{d^2 X}{dx^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y(y)} \frac{d^2 Y}{dy^2}}_{-\beta^2} + \underbrace{\frac{1}{Z(z)} \frac{d^2 Z}{dz^2}}_{\alpha^2 + \beta^2} = 0$$

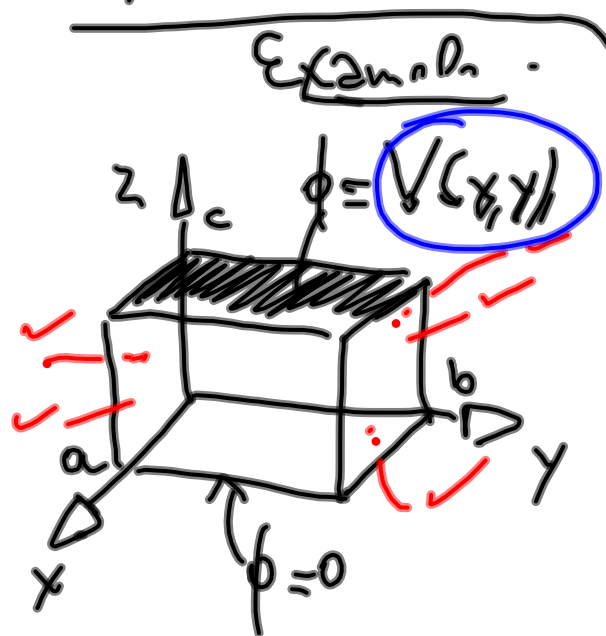
$-\alpha^2$
 $\alpha^2 + \beta^2$
 $-\beta^2$

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\alpha^2,$$

$$X(x) = e^{\pm i\alpha x} \rightarrow \sin \alpha x$$

$$Y(y) = e^{\pm i\beta y} \rightarrow \sin \beta y$$

$$Z(z) = e^{\pm \sqrt{\alpha^2 + \beta^2} z} \rightarrow \sinh(\sqrt{\alpha^2 + \beta^2} z)$$



$$\sin(\alpha x) = 0 \text{ if } x = 0$$

$$\sin(\alpha a) = 0 \rightarrow \alpha_n = \frac{n\pi}{a}$$

$$\sin(\beta a) = 0 \rightarrow \beta_m = \frac{m\pi}{b}$$

$$\phi(x, y, z) = \sum_{n, m} A_{n, m} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\sqrt{\alpha_n^2 + \beta_m^2} z)$$

$z = c$

$$V(x, y) = \sum_{n, m} A_{n, m} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\sqrt{\alpha_n^2 + \beta_m^2} c)$$

give

?