$$W = \boxed{\frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ (i\neq j)}}^{n} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}} .$$

$$W = \boxed{\frac{1}{8\pi\epsilon_0} \int\int \frac{\rho(\vec{x})\,\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x\, d^3x'} = \frac{1}{2}\int \rho(\vec{x})\phi(\vec{x})\, d^3x =$$

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}\,')\, d^3x'}{|\vec{x} - \vec{x}\,'|}$$

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

$$= -\frac{\epsilon_0}{2}\int \phi\, \nabla^2\phi\, d^3x = \frac{\epsilon_0}{2}\int |\nabla\phi|^2 d^3x = \boxed{\frac{\epsilon_0}{2}\int \vec{E}^2 d^3x}$$

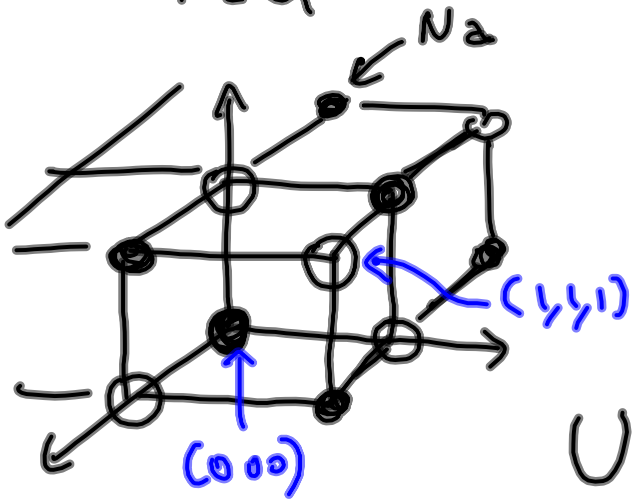
$$\nabla\cdot(\phi\nabla\phi) = |\nabla\phi|^2 + \phi\nabla^2\phi$$

localized charges

$$\int_V \nabla\cdot\vec{A} = \oint_S \vec{A}\cdot\vec{n}\, dS$$

Electrostatic energy of ionic crystals

$Na^{+}Cl^{-}$



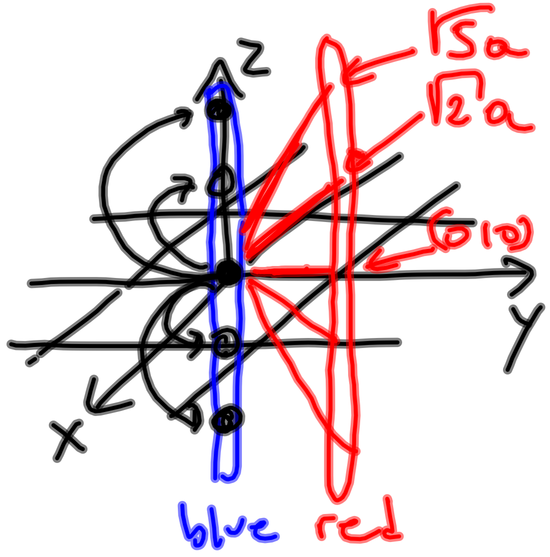
$\vec{n}a = (n_x, n_y, n_z)\,a$, $a \approx 2.8\ \text{Å}$

n_x, n_y, n_z : $-2, -1, 0, 1, 2, \ldots$

$$U = \frac{e^2}{a} \sum_{\vec{n} \neq \vec{0}} (-1)^{n_x+n_y+n_z} \frac{1}{|\vec{n}|}$$

$$\sum_{\vec{n} \neq \vec{0}} = \sum_{n_x}\sum_{n_y}\sum_{n_z}$$

$$\int^{\Lambda} \frac{dx}{x} \sim \ln \Lambda$$



$$U_0 = \frac{2e^2}{a}\overbrace{\left[-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots\right]}^{-\ln 2}$$

$z>0$

$z<0$

- - - -

"Madelung sums"

$$U_0 + 4U_{10} + \cdots$$

N_A

$$\boxed{\underbrace{NU}_{\text{moles}} = \frac{-\underbrace{N}_{\text{moles}}e^2}{a}(1.7476)}$$

2.8 Orthogonal functions and expansions

$\xi \quad (a,b) \quad U_n(\xi)$

orthogonal, square integrable

$$\int_a^b U_n^*(\xi)\, U_m(\xi)\, d\xi = \delta_{nm}$$

$$\int_a^b |U_n(\xi)|^2 d\xi = 1$$

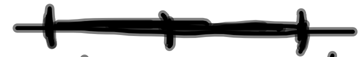
$$f(\xi) = \sum_n a_n U_n(\xi)$$

$$\sum_n U_n^*(\xi') U_n(\xi) = \delta(\xi - \xi')$$

completeness relation

$$\boxed{\int_a^b U_m^* f(\xi)\, d\xi = \sum_n a_n \underbrace{\int_a^b U_m^* U_n(\xi)\, d\xi}_{\delta_{mn}} = a_m}$$

$$\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi m x}{a}\right), \quad \sqrt{\frac{2}{a}}\cos\left(\frac{2\pi m x}{a}\right)$$



$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty}\left[A_m \cos\left(\frac{2\pi m x}{a}\right) + B_m \sin\left(\frac{2\pi m x}{a}\right)\right]$$

$$A_m = \frac{2}{a}\int_{-a/2}^{+a/2} f(x)\cos\left(\frac{2\pi m x}{a}\right)dx$$

B_m sin

$$a \to \infty, \quad \frac{2\pi m}{a} = k, \quad \sum_m \longrightarrow \frac{a}{2\pi}\int_{-\infty}^{+\infty} dk$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{A(k)}_{\text{"}a_n\text{"}} e^{ikx}\, dk$$

sin, cos

sin ± i cos, e^{ikx}, e^{-ikx}

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(k-k')x}\, dx = \delta(k-k')$$

orthogonality

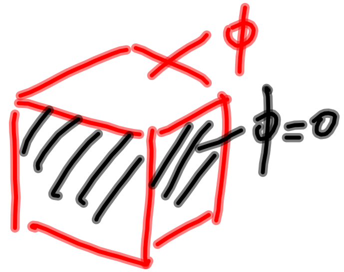
$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-x')}\, dk = \delta(x-x')$$

completeness

2.9 Separation of variables

Laplace eq. $\nabla^2\phi = 0$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$



$$\phi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

and then divide by XYZ

$$\underbrace{\frac{1}{X(x)}\frac{d^2X}{dx^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y(y)}\frac{d^2Y}{dy^2}}_{-\beta^2} + \underbrace{\frac{1}{Z(z)}\frac{d^2Z}{dz^2}}_{\alpha^2+\beta^2} = 0$$

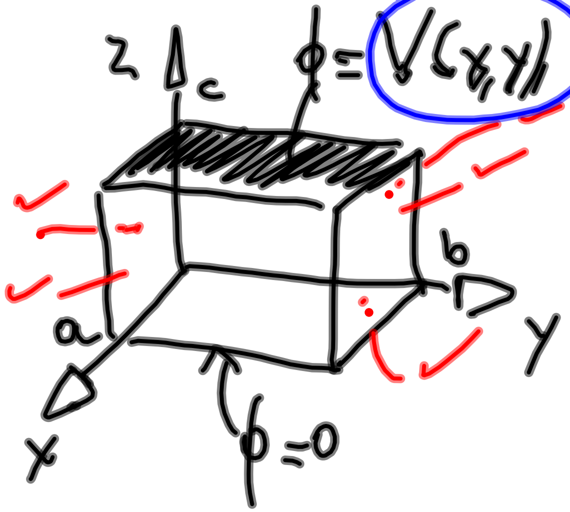
$$-\alpha^2 \qquad\qquad \alpha^2+\beta^2 \qquad\qquad -\beta^2$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2, \quad X(x) = e^{\pm i\alpha x} \rightarrow = \sin\alpha x$$

$$Y(y) = e^{\pm i\beta y} \rightarrow = \sin\beta y$$

$$Z(z) = e^{\pm\sqrt{\alpha^2+\beta^2}\, z} \rightarrow = \sinh\left(\sqrt{\alpha^2+\beta^2}\, z\right)$$

Example.



$\sin(\alpha x) = 0$ if $x = 0$

$\sin(\alpha a) = 0 \longrightarrow \boxed{\alpha_n = \frac{n\pi}{a}}$

$\sin(\beta a) = 0 \longrightarrow \boxed{\beta_m = \frac{m\pi}{b}}$

$$\phi(x,y,z) = \sum_{n,m} A_{n,m} \sin(\alpha_n x) \sin(\beta_m y) \sinh\left(\sqrt{\alpha_n^2+\beta_m^2}\, z\right)$$

$z = c$

$$\boxed{\underbrace{V(x,y)}_{\text{given}} = \sum_{n,m} \underbrace{A_{n,m}}_{?} \sin(\alpha_n x) \sin(\beta_m y) \sinh\left(\sqrt{\alpha_n^2+\beta_m^2}\, c\right)}$$