## 2.8 Orthogonal functions and expansions

$\xi$ ; $(a,b)$ ; $U_n(\xi)$ $(n=1,2,3,\ldots)$

coordinates ; domains

$$\int_a^b U_n^*(\xi)\,U_m(\xi)\,d\xi = \delta_{nm} \quad ; \quad \int_a^b |U_n(\xi)|^2\,d\xi = 1$$

orthogonal ; square integrable

Expansion of an arbitrary function $f(\xi)$ :

analog of $\vec{x} = \sum_{i=1}^{3} c_i \hat{e}_i$ (3D) $\rightarrow$

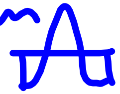
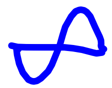
$$f(\xi) = \sum_n a_n U_n(\xi)$$

$\infty$ D

$$\left[\int_a^b U_m^*(\xi) f(\xi)\,d\xi = \sum_n a_n \underbrace{\int_a^b U_m^*(\xi)\,U_n(\xi)\,d\xi}_{\delta_{nm}} = a_m\right]$$

Example $\sqrt{\frac{2}{a}}\,\sin\left(\frac{2\pi m x}{a}\right), \sqrt{\frac{2}{a}}\,\cos\left(\frac{2\pi m x}{a}\right)$


$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos\left(\frac{2\pi m x}{a}\right) + B_m \sin\left(\frac{2\pi m x}{a}\right)$$

even  odd 

$$A_m = \frac{2}{a}\int_{-a/2}^{a/2} f(x)\cos\left(\frac{2\pi m x}{a}\right)dx$$

$B_m =$ ....sin.....

$$a \to \infty, \quad \frac{2\pi m}{a} = k, \quad \sum_{m=1}^{\infty} \to \frac{a}{2\pi}\int dk$$

# Fourier integrals

$$\left[\cos k \pm i \sin k = e^{\pm i k}\right]$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{A(k)}_{\text{analog of } a_n} e^{ikx}\, dk$$

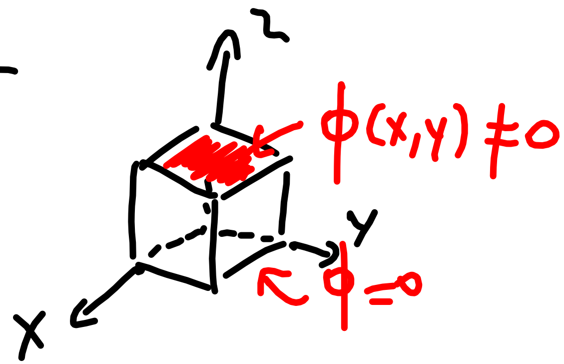
symmetric

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(k-k')x}\, dx = \delta(k-k') \quad \text{orthogonality}$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-x')}\, dk = \delta(x-x') \quad \text{completeness}$$

## 2.9 Separation of variables




$$\nabla^2 \phi = 0 \quad (\rho = 0)$$

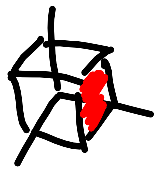
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\phi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

and then divide by $\phi = XYZ$

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-\beta^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\alpha^2+\beta^2} = 0$$

or $\quad -\alpha^2 \quad \alpha^2+\beta^2 \quad -\beta^2$

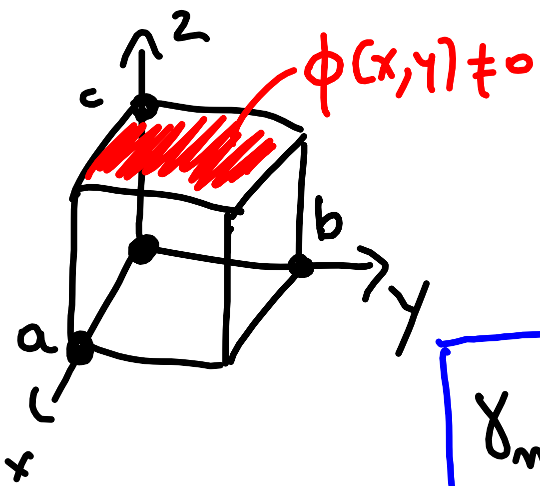


or $\quad \alpha^2+\beta^2 \quad -\alpha^2 \quad -\beta^2$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2 ; \quad X(x) = e^{\pm i\alpha x} \longrightarrow \sin(\alpha x),\ \cos(\alpha x)$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2 ; \quad Y(y) = e^{\pm i\beta y} \longrightarrow \sin(\beta y),\ \cos(\beta y)$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = \alpha^2+\beta^2 ; \quad Z(z) = e^{\pm\sqrt{\alpha^2+\beta^2}\,z} \longrightarrow \sinh\left(\sqrt{\alpha^2+\beta^2}\,z\right),\ \cosh\left(\sqrt{\alpha^2+\beta^2}\,z\right)$$



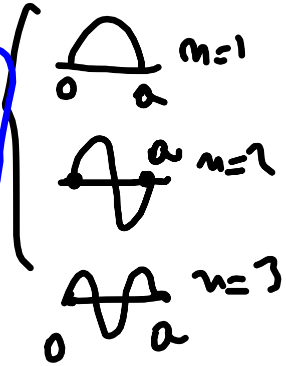

$\sin(\alpha x)$

$\hookrightarrow \sin(\alpha 0) = 0$

$\sin(\alpha a) = 0 \longrightarrow \boxed{\alpha_n = \frac{n\pi}{a}}$

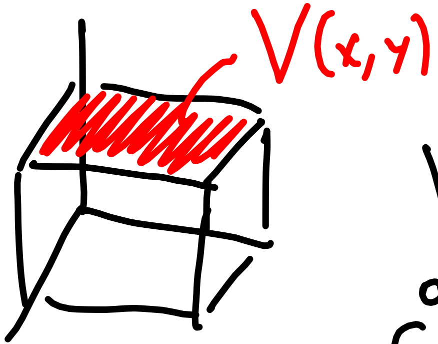
$\sin(\beta y) \longrightarrow \boxed{\beta_m = \frac{m\pi}{b}}$

$$\boxed{\gamma_{nm} = \sqrt{\alpha_n^2+\beta_m^2}}$$



$$\phi(x,y,z) = \sum_{n,m} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)$$

?




fixed

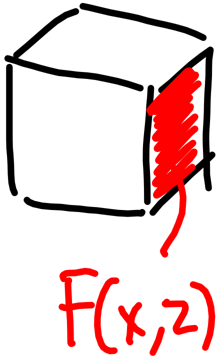
$$V(x,y) = \sum_{nm} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} c)$$

Use: $\frac{2}{a}\int_0^a dx \, \sin(\alpha_n x)\sin(\alpha_m x) = \delta_{nm}$ orthogonality

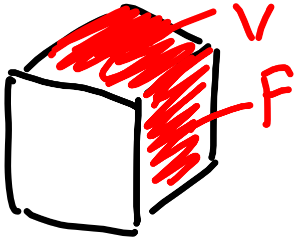
$$\int_0^a\int_0^b \sin(\alpha_\ell x)\sin(\beta_p y)\, V(x,y)\,dx\,dy = \sum_{nm} A_{nm}\int_0^a\int_0^b \sin(\alpha_\ell x)\sin(\beta_p y)\,\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm}c)\,dx\,dy$$

$\delta_{\ell n}$ $\delta_{pm}$


$$\boxed{A_{nm} = \frac{4}{ab\sinh(\gamma_{nm}c)}\int_0^a dx\int_0^b dy\; V(x,y)\sin(\alpha_n x)\sin(\beta_m y)}$$




$$\boxed{B_{nm} = \frac{4}{ac\sinh(\gamma_{nm}b)}\int_0^a dx\int_0^c dz\; F(x,z)\sin(\alpha_n x)\sin(\beta_m z)}$$




$\frac{\pi m}{b}$

$$\phi_1 = \sum_{nm} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)$$

$$\phi_2 = \sum_{nm} B_{nm} \sin(\alpha_n x)\sin(\beta_m z)\sinh(\gamma_{nm} y)$$

$\frac{\pi n}{a}$ $\frac{\pi m}{c}$

$$\phi = \phi_1 + \phi_2$$

Check:

| | $\phi_1$ | $\phi_2$ | $\boxed{\phi_1 + \phi_2}$ |
|---|---|---|---|
| $x = 0$ | 0 | 0 | 0 |
| $y = 0$ | 0 | 0 | 0 |
| $z = 0$ | 0 | 0 | 0 |
| $x = a$ | 0 | 0 | 0 |
| $y = b$ | 0 | F | F |
| $z = c$ | V | 0 | V |