3.1

$$\nabla^2 \Phi = 0$$

$$\frac{\partial}{\partial r} \quad \cdots \quad \frac{\partial}{\partial \theta} \quad \cdots \quad \frac{\partial}{\partial \phi}$$

$$\Phi = \frac{U}{r} P Q$$

$$Q = e^{\pm i m \phi}$$, m = integer

$$\frac{d^2 U}{dr^2} - \frac{l(l+1)U}{r^2} = 0 \quad , \quad \frac{U}{r} = A\frac{r^{l+\cancel{1}}}{\cancel{r}} + B\frac{r^{-l}}{r}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0$$

$x = \cos\theta$

$\underline{m = 0}$

1. $\ell = 0$ or a positive integer

2. $P_0(x) = 1$

   $P_1(x) = x$

   $P_2(x) = \frac{1}{2}(3x^2 - 1)$

   . . . .

$x = \cos\theta$

$dx = -\sin\theta\, d\theta$

3. $$\int_{-1}^{1} P_{\ell'}(x) P_\ell(x)\, dx = \frac{2\delta_{\ell\ell'}}{2\ell+1}$$

3.3 Problems with Azimuthal Symmetry

$m=0$

$$\Phi = \frac{U(r)}{r}\,P(\theta)\,Q \rightsquigarrow 1 =$$

$$= \sum_{l=0}^{\infty}\left(\frac{A_l\, r^{l+1} + B_l\, r^{-l}}{r}\right) P_l(\cos\theta)$$

?

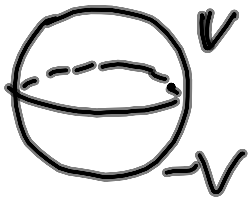
$r<a$: $B_l = 0$

$r=a$:

$$\int_0^\pi d\theta \,\sin\theta\, P_m \quad V(\theta) = \sum_{l=0}^{\infty} A_l\, a^l \underbrace{\int_0^\pi P_l(\cos\theta)\, P_m \sin\theta\, d\theta}_{\text{orthog. relation}}$$

$$\boxed{A_\ell = \frac{2\ell+1}{2a^\ell} \int_0^\pi d\theta \, \sin\theta \, P_\ell(\cos\theta) \, V(\theta)}$$

$V(\theta)$



3.5 $m \neq 0$

$$\ell = 0, 1, 2, 3, \ldots$$

$$m = -\ell, -(\ell-1), \ldots, 0, \ldots, \ell-1, \ell$$

$$P_\ell^m(\cos\theta) \qquad \Phi = \frac{U}{r} P Q$$

$$\boxed{Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi} \cdot \frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi}}$$

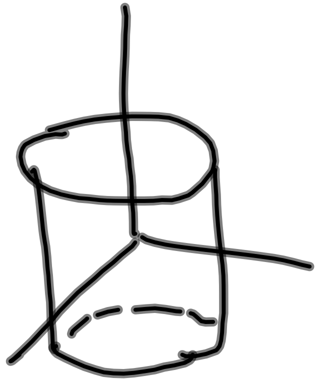
$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\; Y_{\ell' m'}^{*}(\theta,\phi)\, Y_{\ell m}(\theta,\phi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$



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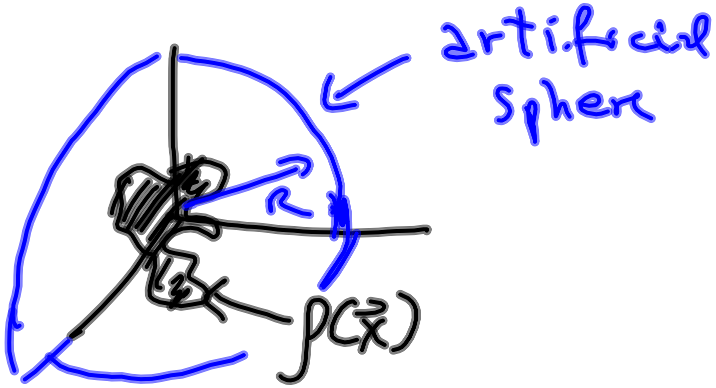
$$\boxed{\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^{l} + B_{lm} r^{-l-1}\right] Y_{lm}(\theta,\phi)}$$

4.1 Multipole Expansion



$$\Phi = \sum_{l}\sum_{m} B_{lm}\, r^{-l-1}\, Y_{lm}(\theta,\phi)$$

$$B_{lm} = \frac{1}{4\pi\epsilon_0}\,\frac{4\pi}{2l+1}\,q_{lm}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$$

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = 4\pi \sum_{l} \sum_{m} \frac{1}{2l+1} \frac{r_<^{l}}{r_>^{l+1}} Y_{lm}^{*}(\theta',\phi') Y_{lm}(\theta,\phi)$$

θ,ϕ θ',ϕ'

$r_<$ = smaller of $|\vec{x}| = r$ and $|\vec{x}\,'| = r'$

$r_>$ = larger of $|\vec{x}|$ and $|\vec{x}\,'|$

outside sphere of Radius R

$$\Phi = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}\,') d^3x' \, 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r'^{l}}{r^{l+1}} Y_{lm}^{*}(\theta',\phi') Y_{lm}(\theta,\phi)$$

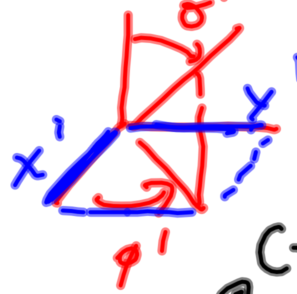
$$\boxed{q_{lm} = \int Y_{lm}^{*}(\theta',\phi') r'^{l} \rho(\vec{x}\,') d^3x'}$$

$$q_{00} = \int \underbrace{Y^*_{00}}_{1/\sqrt{4\pi}} (r')^0 \rho(\vec{x}')\, d^3x' = \frac{1}{4\pi} q$$

$$q = \int \rho(\vec{x}')\, d^3x'$$

$$q_{11} = \int \underbrace{Y^*_{11}(\theta',\phi')}_{-\sqrt{\frac{3}{8\pi}}\,\sin\theta'\, \underbrace{e^{-i\phi'}}_{\cos\phi' - i\sin\phi'}} \; r' \rho(\vec{x}')\, d^3x'$$

$$x' = r' \sin\theta' \cos\phi'$$
$$y' = r' \sin\theta' \sin\phi'$$



$$= -\sqrt{\frac{3}{8\pi}} \int (x' - iy') \rho(\vec{x}')\, d^3x'$$

$$= -\sqrt{\frac{3}{8\pi}} (p_x - i p_y)$$

$$q_{10} = \sqrt{\frac{3}{4\pi}}\, p_z$$

$$\vec{p} \underset{\text{def}}{=} \int \vec{r}\,' \rho(\vec{x}')\, d^3x'$$

(x', y', z')