

$$q_{10} = \sqrt{\frac{3}{4\pi}} P_2 \quad q_{1-1} = -q_{11}^*$$

$$q_{22} \stackrel{\text{def}}{=} \int Y_{22}^*(\theta', \phi') (r')^2 \rho(\vec{x}') d^3x'$$

$$\frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta' e^{-2i\phi'} \quad r'^2 \sin^2 \theta' (\cos \phi' - i \sin \phi')^2$$

$$= \frac{1}{12} \sqrt{\frac{15}{2\pi}} \int d^3x' \rho(\vec{x}') 3 (x'^2 - 2ix'y' - y'^2)$$

$$Q_{ij} = \int (3x'_i x'_j - r'^2 \delta_{ij}) \rho(\vec{x}') d^3x'$$

$x'_1 = x', x'_2 = y', x'_3 = z'$

$$q_{22} = \frac{1}{12} \sqrt{\frac{15}{2\pi}} (Q_{11} - 2iQ_{12} - Q_{22})$$

$$\begin{aligned} \Phi &= \frac{1}{4\pi\epsilon_0} \sum_{l,m} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}}{r^{l+1}} \\ &= \frac{1}{4\pi\epsilon_0} \left[ \underbrace{4\pi q_{00} \frac{Y_{00}}{r}}_{\frac{q}{\sqrt{4\pi}}} + \frac{4\pi}{3} q_{11} \frac{Y_{11}}{r^2} + \frac{4\pi}{3} q_{10} \frac{Y_{10}}{r^2} + \frac{4\pi}{3} q_{1-1} \frac{Y_{1-1}}{r^2} + \dots \right] \end{aligned}$$

$\frac{1}{2} \frac{(p_x - ip_y)(x + iy)}{r^3}$   
 $-\sqrt{\frac{3}{8\pi}} \frac{(p_x - ip_y)}{r^2}$   
 $-\sqrt{\frac{2}{8\pi}} \sin\theta e^{i\phi}$

$$\vec{E} = -\nabla\Phi = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{r} + \frac{\vec{p} \cdot \vec{x}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} + \dots \right]$$

### 4.3 Electrostatics with Ponderable Media

The diagram illustrates the concept of electric polarization in a dielectric medium. It shows a battery connected to a capacitor. In the first part, a dielectric sphere is placed between the capacitor plates, with an external electric field  $E_0$  and a question mark next to  $\vec{E}$ . The second part shows a detailed view of the dielectric sphere, which is filled with dipoles (represented by small circles with '+' and '-' signs). A red box highlights a small volume element  $\Delta v$  within the dielectric, with a red arrow pointing to the polarization vector  $\vec{P}(\vec{x})$  and the text "electric polarization". The third part shows a coordinate system with a red box and vectors  $\vec{x}$  and  $\vec{x}'$ . The equation below the diagram is:

$$\Delta \Phi(\vec{x}, \vec{x}') = \frac{1}{4\pi\epsilon_0} \left[ \frac{\rho(\vec{x}') \Delta v}{|\vec{x} - \vec{x}'|} + \frac{\vec{P}(\vec{x}') \cdot (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \Delta v + \dots \right]$$

The term  $\rho(\vec{x}') \Delta v$  is labeled with a blue bracket and the word "charge".

$$\Phi(\vec{x}) \approx \frac{1}{4\pi\epsilon_0} \int d^3x' \left[ \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\vec{P}(\vec{x}') \cdot (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} \underbrace{\left[ \rho(\vec{x}') - \nabla' \cdot \vec{P}(\vec{x}') \right]}_{\text{effective charge density}}$$

integ. by parts  $\nabla' \left( \frac{1}{|\vec{x}-\vec{x}'|} \right)$   
 $\int \nabla \cdot \left( \frac{1}{r} \vec{P} \right) = 0$

$$\nabla^2 \Phi = \int \underbrace{\nabla^2 \frac{1}{r}}_{-4\pi \delta} [ ] \text{ density}$$

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0} (\rho - \nabla \cdot \vec{P})$$

$$\vec{D} \stackrel{\text{def}}{=} \epsilon_0 \vec{E} + \vec{P} \quad \left| \quad \begin{aligned} \nabla \cdot \vec{D} &= \epsilon_0 \nabla \cdot \vec{E} + \nabla \cdot \vec{P} \\ &= \epsilon_0 \rho + \nabla \cdot \vec{P} \end{aligned} \right.$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0} (\rho - \nabla \cdot \vec{P})$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

↓  
↑ electric susceptibility of the medium

$$\vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}$$

$$= \underbrace{\epsilon_0 (1 + \chi_e)}_{\epsilon} \vec{E}$$

electric permittivity

$$\text{dielectric constant} = \frac{\epsilon}{\epsilon_0}$$

$\chi_e$  is uniform in space

$$\underline{\nabla \cdot \vec{E}} = \frac{1}{\epsilon_0} (\rho - \nabla \cdot \vec{P}) \stackrel{\uparrow}{=} \frac{1}{\epsilon_0} (\rho - \epsilon_0 \chi_e \underline{\nabla \cdot \vec{E}})$$

$\chi_e$  unif. in space

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0(1+\chi_e)} = \frac{\rho}{\epsilon}$$

←

