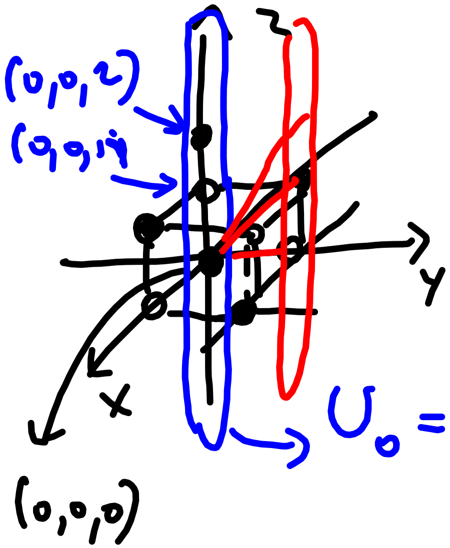

$$U = \frac{e^2}{4\pi\epsilon_0 a} \sum_{n_x} \sum_{n_y} \sum_{n_z} \frac{(-1)^{n_x+n_y+n_z}}{|\vec{n}|} \quad \neq (0,0,0)$$

electrostatic energy of ion at (0,0,0)

$z>0, z<0$

$$U_0 = \frac{2e^2}{4\pi\epsilon_0 a}\Big[\underbrace{-\frac{1}{1} + \frac{(-1)^2}{2} - \frac{1}{3} + \frac{1}{4} - \cdots}_{-\ln 2}\Big]$$

$\sum_n \frac{1}{n}$ D $\qquad \sum_n \frac{1}{n^2}$ C

$$U = U_0 + 4U_{01} + \cdots \cong -\frac{e^2}{4\pi\epsilon_0 a}\underbrace{(1.7476)}_{\text{Madelung constant}}$$

- Energy needed to dissociate crystal into well separated ions
- "Madelung constant" wikipedia

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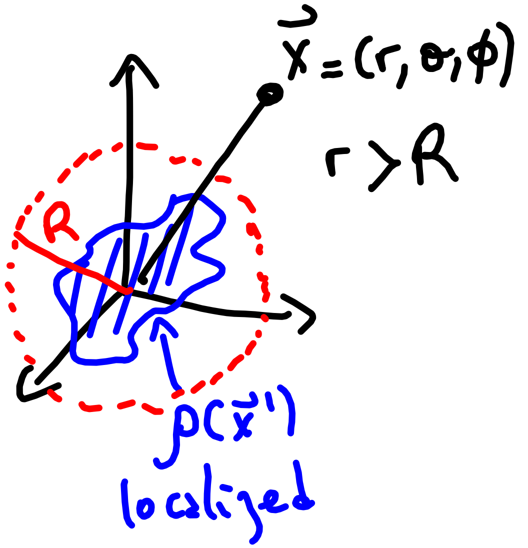
Midterm Given Feb. 12 ; returned Feb. 19
No lecture Feb. 17 → (office hours)

4.1 Multipole expansion



$r > R$

$$\Phi(r,\theta,\varphi) = \sum_{l,m} B_{lm}\, r^{-l-1}\, Y_{lm}(\theta,\phi)$$

$\hookrightarrow B_{lm} = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{2l+1} q_{lm}$ change of notation

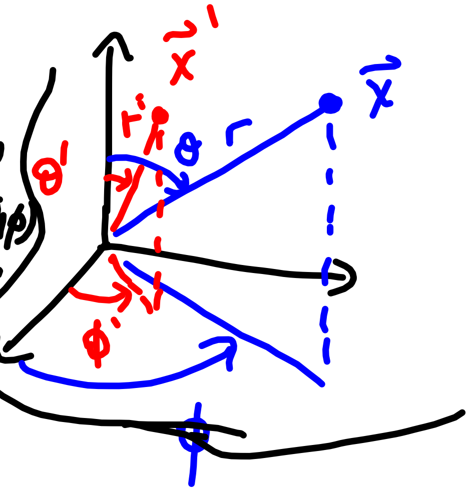
$$\Phi(r,\theta,\varphi) = \frac{1}{4\pi\epsilon_0}\int_R \frac{\rho(\vec{x}\,')\, d^3x'}{|\vec{x}-\vec{x}\,'|}$$

Compare both and find q_{lm}.

"Magic" formula [Eq (3.70)] :

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = \sum_{l,m} \frac{4\pi}{2l+1} \frac{r'^{l}}{r^{l+1}} Y^*_{lm}(\theta',\phi') \, Y_{lm}(\theta,\phi)$$

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \underbrace{\left[\int d^3x'\,\rho(\vec{x}\,')\, r'^{l}\, Y^*_{lm}(\theta',\phi')\right]}_{q_{lm}} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$



$$q_{00} = \int d^3x'\,\rho(\vec{x}\,')\,(r')^0 \underbrace{Y^*_{00}(\theta',\phi')}_{\frac{1}{\sqrt{4\pi}}} = \frac{1}{\sqrt{4\pi}} \underbrace{\int d^3x'\,\rho(\vec{x}\,')}_{q} = \frac{q}{\sqrt{4\pi}}.$$

charge

$$q_{11} = \int d^3x' \, \rho(\vec{x}') \, (r')^1 \, Y^*_{11}(\theta', \phi')$$

$$Y^*_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta' \, e^{-i\phi'}, \qquad e^{-i\phi'} = \cos\phi' - i \sin\phi'$$

$$x' = r' \sin\theta' \cos\phi'$$

$$y' = r' \sin\theta' \sin\phi'$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int d^3x' \, \rho(\vec{x}') \, (x' - i y')$$

electric dipole moment

$$\vec{p} \underset{\text{def}}{=} \int d^3x' \, \rho(\vec{x}') \, \vec{r}'$$

$\vec{r}' = (x', y', z')$

$$\boxed{q_{11} = -\sqrt{\frac{3}{8\pi}} \, (p_x - i p_y)}$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} \, p_z$$

. . . .

$$q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int d^3x' \rho(\vec{x}') \left(x'^2 - 2i\,x'y' - y'^2 \right)$$

q_{21}, q_{20}, q_{2-1}, q_{2-2} ....

Quadrupole moment tensor, 3x3 matrix, traceless

$$Q_{11} = \int d^3x' \rho(\vec{x}') \left(3x'^2 - r'^2 \right)$$

$r'^2 = x'^2 + y'^2 + z'^2$

$$Q_{ij} = \int d^3x' \rho(\vec{x}') \left(3x_i' x_j' - r'^2 \delta_{ij} \right)$$

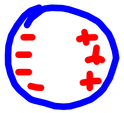
$i, j = 1, 2, 3$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{ij} Q_{ij}\frac{x_i x_j}{r^5} + \cdots \right]$$

4.3 Electrostatic with ponderable media

Previous example

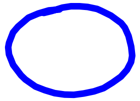
$\longrightarrow \vec{E}_0$



electric dipole was induced

Atoms, molecules are like tiny metallic spheres

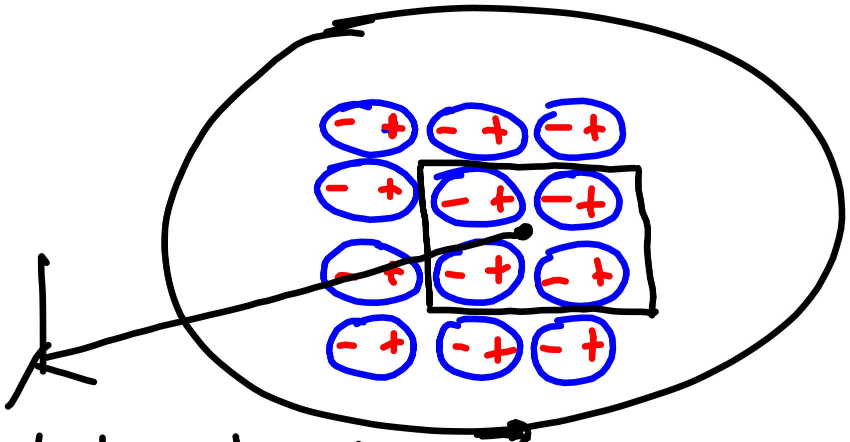
$\vec{E}_0 = 0$



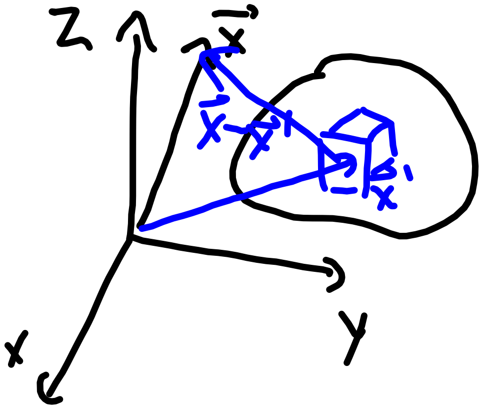
$\vec{p} = 0$

$\vec{E}_0 \neq 0$

$\vec{p} \neq 0$



electric polarization $\vec{P}(\vec{x})$



$$\Delta\Phi_{\text{from } \Delta V} \approx \frac{1}{4\pi\epsilon_0}\left[\frac{\rho(\vec x')\,\Delta V}{|\vec x-\vec x'|} + \frac{\vec P(\vec x')\cdot(\vec x-\vec x')}{|\vec x-\vec x'|^3}\Delta V\right]$$

$$\frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\vec p\cdot\vec x}{r^3} + \cdots\right]$$

$$\Phi(\vec x) = \frac{1}{4\pi\epsilon_0}\int d^3x'\left[\frac{\rho(\vec x')}{|\vec x-\vec x'|} + \frac{\vec P(\vec x')\cdot(\vec x-\vec x')}{|\vec x-\vec x'|^3}\right]$$

$$\rightarrow \nabla'\left(\frac{1}{|\vec x-\vec x'|}\right)$$