Poisson distribution and application

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The Poisson distribution is one of three discrete distributions, Binomial, Poisson, and Hypergeometric, that use integers as random variables. The Poisson distribution was discovered in 1838 by Simeon-Denis Poisson as an approximation to the binomial distribution, when the probability of success is small and the number of trials is large. The Poisson distribution is called the law of small numbers because Poisson events occur rarely even though there are many opportunities for these evens to occur. In this paper, the Poisson distribution is discussed. Examples are given to show that The Poisson distribution can be a good approximation in many cases.

In probability theory and statistics, the Poisson distribution is a discrete probability distribution that expresses the probability of a number of events occurring in a fixed period of time if these events occur with a known average rate and independently of the time since the last event. The Poisson distribution can be applied to systems with a large number of possible outcomes, each of which is rare. The Poisson distribution can also be used for the number of events in other specified intervals such as distance, area or volume.

The distribution was discovered by Simon Denis Poisson (1781-1840) and published together with his probability theory, in 1838 in his work Research on the Probability of Judgments in Criminal and Civil Matters[1]. The work focused on certain random variables N that count, among other things, a number of discrete occurrences, or sometimes it is called "arrivals" that take place during a time-interval of given length. The Poisson distribution equation is obtained later in this paper by giving a classic example. Now, we just take a brief look at the equation. If the expected number of occurrences in this interval is λ, then the probability that there are exactly k occurrences is equal to

$$f(k,\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2... \quad (1)$$

where k is a non-negative integer, $k = 0, 1, 2, ...$ denotes the number of occurrences of an event - the probability of which is given by the function. λ is a positive real number, equal to the expected number of occurrences that occur during the given interval. For example, if the events occur on average 2 times per minute, and you are interested in the number of events occurring in a 20 minute interval, you would use as model a Poisson distribution with $\lambda = 20 * 2 = 40$. Fig.1 shows the Poisson distribution for different value of λ. As a function of k, this is the probability mass function, which means it is a function that gives the probability that a discrete random variable is exactly equal to some value.

A classic example of Poisson distribution is the nuclear decay of atoms[2]. The decay of a radioactive sample is a case in point because, once a particle decays, it does not decay again. If the observation time dt is small enough

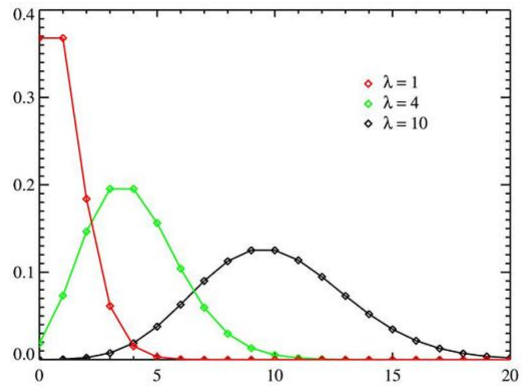

FIG. 1: The Poisson distribution. The horizontal axis is the index k. The function is only defined at integer values of k (empty lozenges). The connecting lines are only guides for the eye.

so that the emission of two or more particles is negligible, then the probability that one particle is emitted is μdt with constant μ and $\mu dt << 1$. We can set up a recursion relation for the probability $P_n(t)$ of observing n counts during a time interval t. For $n > 0$ the probability $P_n(t+dt)$ is composed of two mutually exclusive events that (1) n particles are emitted in the time t, none in dt, and (2) $n-1$ particles are emitted in time t, one in dt. Therefore

$$P_n(t+dt) = P_n(t)P_0(dt) + P_{n-1}(t)P_1(dt) \quad (2)$$

Here we substitute the probability of observing one particle, $P_1(dt) = \mu dt$, and no particle, $P_0(dt) = 1 - P_1(dt) = 1 - \mu dt$, in time dt. This yields

$$P_n(t+dt) = P_n(t)(1-\mu dt) + P_{n-1}(t)\mu dt \quad (3)$$

So, after rearranging and dividing by dt, we get

$$\frac{dP_n(t)}{dt} = \frac{dP_n(t+dt) - P_n(t)}{dt} = \mu dP_{n-1}(t) - \mu P_n(t) \quad (4)$$

For $n = 0$ this differential recursion relation simplifies, because there is no particle in times t and dt giving

$$\frac{dP_0(t)}{dt} = -\mu P_0(t) \quad (5)$$

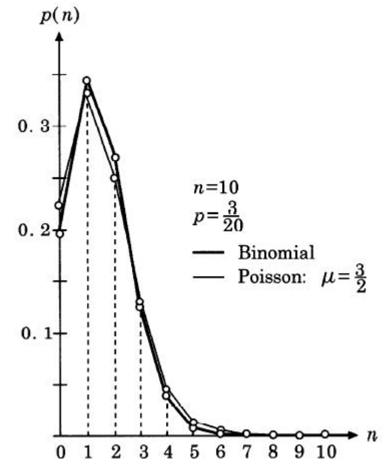

FIG. 2: Poisson distribution compared with binomial distribution[2].

The ODE says that particles have a constant decay probability and decay removes them from the distribution. This ODE integrates to $P_0(t) = e^{-\mu t}$ if the probability that no particle is emitted during a zero time interval $P_0(0) = 1$ is used. Here $P_0(0) = 1$ means no decay takes place at $t \leq 0$. Back to Eq.(2) for $n = 1$,

$$\dot{P}_1 = \mu(e^{-\mu t} - P_1), P_1(0) = 0 \tag{6}$$

and solve the homogeneous equation, which is the same for P_1 as Eq.(5). This yields$P_1 = \mu_1 e^{-\mu t}$. Then we solve the inhomogeneous Eq.(6) by varying the constant μ_1 to find $\dot{\mu_1} = \mu$, so $P_1 = \mu t e^{-\mu t}$. The general solution is

$$P_n(t) = \frac{(\mu t)^n}{n!} e^{-\mu t} \tag{7}$$

This is an example of the Poisson distribution.

A Poisson distribution becomes a good approximation of the binomial distribution for a large number n of trials and small probability $p \sim \mu/n$, μ a constant[2]. In the limit $n \to \infty$ and $p \to 0$ so that the mean value $np \to \mu$ stays finite, the binomial distribution becomes a Poisson distribution. To prove this theorem, we can apply stirling's formula $n! \sim \sqrt{2\pi n}(n/e)^n$ for large n to the factorials in binomial probability distribution,

$$f(x) = \frac{n!}{x!(n-x)!} p^x q^{n-x} \tag{8}$$

here $q = 1 - p$, keeping x finite while $n \to \infty$. This yields for $n \to \infty$:

$$\begin{aligned} \frac{n!}{(n-x)!} &\sim (\frac{n}{e})^n (\frac{e}{n-x})^{n-x} \sim (\frac{n}{e})^x (\frac{n}{n-x})^{n-x} \\ &\sim (\frac{n}{e})^x (1 + \frac{n}{n-x})^{n-x} \sim (\frac{n}{e})^x e^x \sim n^x \end{aligned} \tag{9}$$

and for $n \to \infty$, $p \to 0$, with $np \to \mu$:

$$(1-p)^{n-x} \sim (1 - \frac{pn}{n})^n \sim (1 - \frac{\mu}{n})^n \sim e^{-\mu} \tag{10}$$

Finally, $p^x n^x \to \mu^x$, so altogether

$$\begin{aligned} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \to \frac{n^{xp^x}}{x!} (1-p)^{n-x} \to \\ \frac{\mu^x}{x!} e^{-\mu}, n \to \infty, \end{aligned} \tag{11}$$

which we can see that is a Poisson distribution for the random variable $k = x$ with $0 \leq x < \infty$. This limit theorem is a particular example of the laws of large numbers. It is a theorem in probability that describes the long-term stability of the mean of a random variable. Accordingly the Poisson distribution is sometimes called the law of small numbers because it is the probability distribution of the number of occurrences of an event that happens rarely but has very many opportunities to happen[3].

In Eq(1), the random variable k is discrete. The probabilities are properly normalized because $e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1$. The mean value and variance,

$$< k >= e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} = \lambda, \tag{12}$$

$$\sigma^2 =< k^2 > - < k >^2 = \lambda(\lambda + 1) - \lambda^2 = \lambda \tag{13}$$

follow from the characteristic function

$$< e^{itk} >= \sum_{k=0}^{\infty} e^{itk} - \lambda \frac{(\lambda e^{it})^k}{k!} = e^{\lambda(e^{it}-1)} \tag{14}$$

by differentiation and setting $t = 0$, and using that $\sigma^2 =< k^2 > - < k >^2$. From Eq(12)and Eq(13). We can see that the parameter λ in the Poisson distribution is not only the mean number of occurrences$< k >$, but also its variance $\sigma^2 =< k^2 > - < k >^2$. Thus, the number of observed occurrences fluctuates about its mean λ with a standard deviation $\sigma = \sqrt{\lambda}$. These fluctuations are denoted as Poisson noise or (particularly in electronics) as shot noise[4]. The correlation of the mean and standard deviation in counting independent, discrete occurrences is useful scientifically. By monitoring how the fluctuations vary with the mean signal, one can estimate the contribution of a single occurrence, even if that contribution is too small to be detected directly. For example, the charge e on an electron can be estimated by correlating the magnitude of an electric current with its shot noise. If N electrons pass a point in a given time t on the average, the mean current is $I = eN/t$; since the current fluctuations should be of the order $\sigma_I = e\sqrt{N/t}$ (i.e. the variance of the Poisson process), the charge e can be estimated from the ratio σ_I^2/I. An everyday example is the graininess that appears as photographs are enlarged; the graininess is due to Poisson fluctuations in the number of reduced silver grains, not to the individual grains themselves. By correlating the graininess with the degree of enlargement, one can estimate the contribution of an individual grain which is otherwise too small to be seen unaided. Many other molecular applications of Poisson noise have been developed, for example, estimating the number density of receptor molecules in a cell membrane.

Another application for the Poisson distribution is determining the number of spare line replacement units(LRU) that should be initially available to ensure a preselected probability that a spare is available[5]. Frederick J. O'Neal of Bell Laboratories developed such a sparing equation in the 1970s for electronic systems. In his 1989 RAMS article, Al Myrick modified this equation to assure availability for a desired confidence level.

$$CL \leq \sum_{k=0}^{s} \frac{(n\lambda R)^k e^{-(n\lambda R)}}{k!} \tag{15}$$

where n is the number of LRU in service, λ is the failure rate per hour, R is the repair time in hours, CL is the confidence level, and S is the minimum number of spares required. An example of how this equation can be used to determine the number of spares that must be available, taking into account the repair time, the failure rate of the LRU, number of units in service, and a given level of confidence follows. Assume that 2000 LRUs are in service. The failure rate is 121.7 failures per million hours and the mean time to repair a failure is 4 hours. The plan is to have one spare unit available. We want to be 90 present confident that one spare unit will be adequate. Using the equation (15), we can test the hypothesis that one spare will assure 90 present confidence.

$$0.90 \leq \sum_{k=0}^{1} \frac{((2000)(121.7/10^6)(4))^k e^{-[(2000)(121.7/10^6)(4)]}}{k!} \tag{16}$$

$$0.90 \leq 0.3777 + 0.3677 = 0.7454 \tag{17}$$

So, if two spares were made available k=2, then the Confidence Level would beyond 0.9.

Poisson distribution is discussed in this paper. By giving the examples of the nuclear decay of atoms, we can see that the Poisson distribution can be expressed to show the probability of the atom decay. And it also can be used in many other application such as shot noise analyze and LRU determination. Some property of the Poisson distribution can be seen through the discussion.

[1] Irving John Good. Some statistical applications of poisson's work. *Statistical Science*, 1(2):157, 1986.

[2] George Arfken. *Mathematical Methods for Physicists*. 6th edition, 2005.

[3] Ladislaus Bortkiewicz. *The Law of Small Numbers*. 1898.

[4] Winfield Hill. *The Art of Electronics*. 2nd edition, 1989.

[5] Edward R. Sherwin. Applications of poisson distribution. *RAC START Sheet*, 9(2).