Green's Functions and their applications in Physics

Erik M. Olsen*
University of Tennessee
Knoxville, TN 37996-1200
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Differential equations appear frequently in various areas of mathematics and physics. In this paper the method of Green's Functions as solutions to these equations will be discussed in length. Also included will be various examples of physical problems where Green's Functions solutions are useful.

1. INTRODUCTION

In studying physics, it would be difficult to avoid the presence of differential equations. These mathematical constructs appear quite often and can be extremely varied in type. For example, there is the equation of simple harmonic motion (a homogeneous ordinary differential equation):

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \tag{1}$$

There is also Poisson's equation (an inhomogeneous partial differential equation):

$$\nabla^2\psi = -\frac{\rho}{\epsilon_0} \tag{2}$$

Fortunately, there exist many ways, both analytical and numerical, of solving these equations and others.

The method of Green's Functions (named for English mathematician and physicist George Green) is particularly useful for the latter type of equation shown here. In section 2 of this paper the general process of forming a Green's Function and the properties of Green's Functions will be discussed. In section 3 an example will be shown where Green's Function will be used to calculate the electrostatic potential of a specified charge density. In section 4 an example will be shown to illustrate the usefulness of Green's Functions in quantum scattering. Finally, in section 5 a conclusion of all things discussed will be given.

2. GREEN'S FUNCTIONS SOLUTIONS

Suppose we have a differential equation of the following form:

$$Wm(\vec{r}) = f(\vec{r}) \tag{3}$$

Where $m(\vec{r})$ is the function to be determined, $f(\vec{r})$ is a term that contains m and derivatives of m and W is a linear operator. We also assume that this equation is subjected to certain boundary conditions.

Now consider the idea that we can find a function $G(\vec{r}, \vec{r_0})$ that solves this particular differential equation with a delta function as a source (or inhomogeneity) instead of $f(\vec{r})$:

$$WG(\vec{r}, \vec{r_0}) = \delta(\vec{r} - \vec{r_0}) \tag{4}$$

As an example, let $W = \nabla^2$. Using Green's Theorem [1] on $G(\vec{r}, \vec{r_0})$ and $m(\vec{r})$:

$$\int_V (m\nabla^2 G - G\nabla^2 m)d\tau = \int_S (m\nabla G - G\nabla m) \cdot d\vec{A} \tag{5}$$

From the equations above:

$$\int_V [m(\vec{r})\delta(\vec{r} - \vec{r_0}) - G(\vec{r}, \vec{r_0})f(\vec{r})]d\tau \tag{6}$$

$$= \int_S (m(\vec{r})\nabla G(\vec{r}, \vec{r_0}) - G(\vec{r}, \vec{r_0})\nabla m(\vec{r})) \cdot d\vec{A} \tag{7}$$

If we interchange $\vec{r}$ and $\vec{r_0}$ (to be justified later):

$$\int_V [m(\vec{r_0})\delta(\vec{r_0} - \vec{r}) - G(\vec{r_0}, \vec{r})f(\vec{r_0})]d\tau \tag{8}$$

$$= \int_S (m(\vec{r})\nabla G(\vec{r}, \vec{r_0}) - G(\vec{r}, \vec{r_0})\nabla m(\vec{r})) \cdot d\vec{A} \tag{9}$$

By performing the integral on the delta function and adding we find [2]:

$$m(\vec{r}) = \int_V G(\vec{r}, \vec{r_0})f(\vec{r_0})d\tau \tag{10}$$

$$+ \int_S (m(\vec{r_0})\nabla G(\vec{r}, \vec{r_0}) - G(\vec{r}, \vec{r_0})\nabla m(\vec{r_0})) \tag{11}$$

We call $G(\vec{r})$ the **Green's Function** of this particular differential equation.

If we can find a Green's Function $G(\vec{r})$ that satisfies the equation above, then we can find our desired function. The whole problem then is to manipulate this formula to give something useful, and such manipulations are specific to the particular equation to be solved and the geometry of the problem.

The derivation above is not the only way to solve for the Green's Function; it just happened to be the most

*Electronic address: eolsen1@utk.edu

convenient since the linear differential operator was the Laplacian.

Green's Functions are always the solution of a δ-like inhomogeneity. However, it is worthwhile to mention that since the Delta Function is a distribution and not a function, Green's Functions are not required to be functions.

It is important to state that Green's Functions are unique for each geometry. However, you may add a factor $G_0(\vec{r})$ to the Green's Function $G(\vec{r})$ where $G_0(\vec{r})$ satisfies the homogeneous differential equation in question[3]:

$$WG_0(\vec{r}) = 0 \tag{12}$$

Where W is a linear differential operator. They also rely heavily on the specified boundary conditions; a Green's Function for one boundary may not be defined on other.

One of the most useful properties of Green's Functions is that they are always symmetric[4]:

$$G(a,b) = G(b,a) a \neq b \tag{13}$$

When calculating the Green's Function earlier, we depended on switching $\vec{r}$ with $\vec{r_0}$; if the variables of this Green's Function were not symmetric we could not do this. As such, it is worthwhile to prove the symmetry of the variables in any Green's Function.

Suppose we have two Green's Functions $G(\vec{r},\vec{r_1})$ and $G(\vec{r},\vec{r_2})$. We require that these equations satisfy the following equations [5]:

$$\nabla \cdot [p(\vec{r})\nabla G(\vec{r},\vec{r_1})] + q(\vec{r})G(\vec{r},\vec{r_1}) = -\delta(\vec{r}-\vec{r_1}) \tag{14}$$

$$\nabla \cdot [p(\vec{r})\nabla G(\vec{r},\vec{r_2})] + q(\vec{r})G(\vec{r},\vec{r_2}) = -\delta(\vec{r}-\vec{r_2}) \tag{15}$$

Here $p(\vec{r})$ and $q(\vec{r})$ are random functions of $\vec{r}$. We shall specify the Green's Functions further by imposing Dirichlet boundary conditions [5] on them where $G(\vec{r},\vec{r_1})$ and $G(\vec{r},\vec{r_2})$ will yield the same values over the surface S of some volume. Should this condition not be met, the Green's Functions will disappear on S.

Multiplying the equation for $G(\vec{r},\vec{r_1})$ by $G(\vec{r},\vec{r_2})$ and vice versa we find:

$$G(\vec{r},\vec{r_2})\nabla \cdot [p(\vec{r})\nabla G(\vec{r},\vec{r_1})] + \tag{16}$$
$$G(\vec{r},\vec{r_2})q(\vec{r})G(\vec{r},\vec{r_1}) = -G(\vec{r},\vec{r_2})\delta(\vec{r}-\vec{r_1}) \tag{17}$$

$$G(\vec{r},\vec{r_1})\nabla \cdot [p(\vec{r})\nabla G(\vec{r},\vec{r_2})] + \tag{18}$$
$$G(\vec{r},\vec{r_1})q(\vec{r})G(\vec{r},\vec{r_2}) = -G(\vec{r},\vec{r_1})\delta(\vec{r}-\vec{r_2}) \tag{19}$$

Subtracting the second equation from the first:

$$G(\vec{r},\vec{r_2})\nabla \cdot [p(\vec{r})\nabla G(\vec{r},\vec{r_1})] - \tag{20}$$
$$G(\vec{r},\vec{r_1})\nabla \cdot [p(\vec{r})\nabla G(\vec{r},\vec{r_2})] = \tag{21}$$
$$-G(\vec{r},\vec{r_2})\delta(\vec{r}-\vec{r_1}) + G(\vec{r},\vec{r_1})\delta(\vec{r}-\vec{r_2}) \tag{22}$$

Using the following product rule [1]

$$\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla f \tag{23}$$

we can change some terms in our equation:

$$\nabla \cdot [G(\vec{r},\vec{r_2})p(\vec{r})\nabla G(\vec{r},\vec{r_1})] \tag{24}$$
$$-\nabla \cdot [G(\vec{r},\vec{r_1})p(\vec{r})\nabla G(\vec{r},\vec{r_2})] \tag{25}$$
$$= -G(\vec{r},\vec{r_2})\delta(\vec{r}-\vec{r_1}) + G(\vec{r},\vec{r_1})\delta(\vec{r}-\vec{r_2}) \tag{26}$$

Performing a volume integral on both sides and using the Divergence Theorem [1] to simplify the left side:

$$\int_S [G(\vec{r},\vec{r_2})p(\vec{r})\nabla G(\vec{r},\vec{r_1}) \tag{27}$$
$$-G(\vec{r},\vec{r_1})p(\vec{r})\nabla G(\vec{r},\vec{r_2})] \cdot d\vec{A} \tag{28}$$
$$= -G(\vec{r_1},\vec{r_2}) + G(\vec{r_2},\vec{r_1}) \tag{29}$$

By the boundary conditions imposed at the beginning, the left hand side is zero. Hence:

$$G(\vec{r_1},\vec{r_2}) = G(\vec{r_2},\vec{r_1}) \tag{30}$$

Thus proving the symmetry of Green's Function.

3. EXAMPLE: ELECTROSTATIC POTENTIAL[5]

Let us start with Poisson's equation:

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \tag{31}$$

Suppose we have a collection of point charges q_k. From the study of electrostatics, we know the electric potential of such a configuration:

$$\phi = \frac{1}{4\pi\epsilon_0}\sum_k \frac{q_k}{r_k} \tag{32}$$

If instead of a discrete number of point charges we had a continuous distribution of charge (let ρ be such a charge density) then we have:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'})}{|\vec{r}-\vec{r'}|} dV' \tag{33}$$

Where $\vec{r}$ is the vector pointing from the origin to the field point, $\vec{r'}$ is the vector pointing from the origin to the source point and dV' is a volume element.

Observing Poisson's Equation and the equation directly above, we are now in a position to try and find a Green's Function $G(\vec{r},\vec{r'})$. Using the formalism from section 2:

$$\nabla^2 G(\vec{r},\vec{r'}) = -\delta(\vec{r}-\vec{r'}) \tag{34}$$

We will now solve for $G(\vec{r},\vec{r'})$. Employing Green's Identity [1]:

$$\int_V (\phi(\vec{r'})\nabla^2 G(\vec{r},\vec{r'}) - G(\vec{r},\vec{r'})\nabla^2\phi(\vec{r'}))dV' \tag{35}$$
$$= \int_S (\phi(\vec{r'})\nabla G(\vec{r},\vec{r'}) - G(\vec{r},\vec{r'})\nabla\phi(\vec{r'})) \cdot d\vec{A} \tag{36}$$

Where dA is the surface element for the surface that surrounds the volume V.

We shall consider the case where the surface term is zero. This will simplify the problem:

$$\int_V (\phi(\vec{r'})\nabla^2 G(\vec{r},\vec{r'}) - G(\vec{r},\vec{r'})\nabla^2\phi(\vec{r'}))dV' = 0 \quad (37)$$

$$\int_V \phi(\vec{r'})\nabla^2 G(\vec{r},\vec{r'})dV' = \int_V G(\vec{r},\vec{r'})\nabla^2\phi(\vec{r'})dV' \quad (38)$$

Plugging in for the values of $\nabla^2 G(\vec{r},\vec{r'})$ and $nabla^2\phi(\vec{r})$:

$$-\int \phi(\vec{r'})\delta(\vec{r}-\vec{r'})dV' = -\int \frac{G(\vec{r},\vec{r'})\rho(\vec{r'})}{\epsilon_0}dV' \quad (39)$$

So we find:

$$\phi(\vec{r}) = \frac{1}{\epsilon_0}\int G(\vec{r},\vec{r'})\rho(\vec{r'})dV' \quad (40)$$

We are now in a position to find our Green's Function. From studies of Poisson's Equation and Coulomb's Law:

$$\nabla^2\frac{1}{r} = -4\pi\delta(r) \quad (41)$$

Looking back at the original equation for $\nabla^2 G$, we can conclude:

$$G(\vec{r},\vec{r'}) = \frac{1}{4\pi|\vec{r}-\vec{r'}|} \quad (42)$$

This yields for ϕ:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'})}{|\vec{r}-\vec{r'}|}dV' \quad (43)$$

Even though we knew the answer from the very beginning, it is nice to see that the Green's Function formalism gives the correct results. Of course, there are other examples where we don't know the answer from hindsight and must rely on Green's Functions to solve the problem.

4. QUANTUM SCATTERING

Suppose we have a beam of particles incident on a target (represented by a potential $V(\vec{r})$). The particles that hit the target scatter off of it as spherical waves; we represent those waves with the wave function $\psi(\vec{r})$. Obviously $\psi(\vec{r})$ obeys the Schrodinger Equation [5]:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) + V(\vec{r})\psi(\vec{r}) = E\psi(\vec{r}) \quad (44)$$

This will be more useful in the form of the Helmholtz Equation. As such, we define $k^2 = \frac{2mE}{\hbar^2}$ and write:

$$\nabla^2\psi(\vec{r}) + k^2\psi(\vec{r}) = -[-\frac{2mV(\vec{r})}{\hbar}\psi(\vec{r})] \quad (45)$$

Ultimately, we will want a solution that contains the incident wave $e^{i\vec{k_0}\vec{r}}$, the scattered wave $\frac{e^{ikr}}{r}$ and the amplitude of the scattered wave $f_k(\theta,\phi)$. Thus, $\psi(\vec{r})$ will have the asymptotic form:

$$\psi(\vec{r}) \approx e^{i\vec{k_0}\cdot\vec{r}} + f_k(\theta,\phi)\frac{e^{ikr}}{r} \quad (46)$$

Here k_0 is the vector pointing in the direction of the incident wave and k is the vector pointing in the direction of the scattered wave.

By the general form of the Green's Function found in section 2:

$$\psi(r_1) = -\int_V \frac{2m}{\hbar^2}V(\vec{r_2})\psi(\vec{r_2})G(\vec{r_1},\vec{r_2})d^3r_2 \quad (47)$$

Because this is meant to describe an outgoing wave that approaches infinity, we set the surface term equal to zero.

We add a term to make $\psi(\vec{r})$ asymptotic:

$$\psi(\vec{r_1}) = e^{i\vec{k_0}\cdot\vec{r_1}} - \int_V \frac{2m}{\hbar^2}V(\vec{r_2})\psi(\vec{r_2})G(\vec{r_1},\vec{r_2})d^3r_2 \quad (48)$$

Since we are working with the Helmholtz operator, our Green's Function will have the form [5]:

$$G(\vec{r_1},\vec{r_2}) = \frac{e^{ik|\vec{r_1}-\vec{r_2}|}}{4\pi|\vec{r_1}-\vec{r_2}|} \quad (49)$$

Hence we get the exact equation:

$$\psi(\vec{r_1}) = e^{i\vec{k_0}\cdot\vec{r_1}} - \int_V \frac{2m}{\hbar^2}V(\vec{r_2})\psi(\vec{r_2})\frac{e^{ik|\vec{r_1}-\vec{r_2}|}}{4\pi|\vec{r_1}-\vec{r_2}|}d^3r_2 \quad (50)$$

By making approximations on ψ, we can get extremely useful information out of this equation. For example: if we assume the incident wave is not considerably changed by the potential:

$$\psi(\vec{r_2}) = e^{i\vec{k_0}r_2} \quad (51)$$

Plugging this into the integral:

$$\psi_1(\vec{r_1}) = e^{i\vec{k_0}\cdot\vec{r_1}} - \int_V \frac{2m}{\hbar^2}V(\vec{r_2})e^{i\vec{k_0}r_2}\frac{e^{ik|\vec{r_1}-\vec{r_2}|}}{4\pi|\vec{r_1}-\vec{r_2}}d^3r_2 \quad (52)$$

The above equation is known as the Born Approximation; it is very useful for scattering problems where the potential is weak compared to the given potential.

5. CONCLUSION

There are numerous methods available to solve differential equations. At first glance, it may seem as though the method of Green's Functions is rather limited since it can only be used on equations of a particular form and by the fact that not all linear operators admit a Green's Function. Nevertheless, these kinds of equations do appear frequently in physics, so Green's Functions prove to be invaluable in the understanding of physical systems.

[1] D. J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Inc, 1999), 3rd ed.
[2] L. Challis and F. Sheard, *The Green of Green's Functions*, vol. 56 (2003).
[3] D. J. Griffiths, *Introduction to Quantum Mechanics* (Pearson Prentice Hall, 1995), 2nd ed.
[4] W. Strauss, *Partial Differential Equations* (John Wiley and Sons, 1992), 1st ed.
[5] G. Arfken and H. Weber, *Mathematical Methods for Physicists* (Elsevier Adademic Press, 2005), sixth ed.