Exploring Stokes' Theorem

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Stokes' Theorem is widely used in both math and science, particularly physics and chemistry. From the scientific contributions of George Green, William Thompson, and George Stokes, Stokes' Theorem was developed at Cambridge University in the late 1800s. It is based heavily on Green's Theorem which relates a line integral around a closed path to a plane region bound by this path. Stokes' Theorem is identical to Green's Theorem, except one is working with a surface in three dimensions instead of a plane in two dimensions. Stokes' Theorem relates a surface integral to a line integral around the boundary of that surface. Stokes' Theorem can be used to derive several main equations in physics including the Maxwell-Faraday equation, and Ampere's Law.

I. INTRODUCTION

Sir George Gabriel Stokes' name was given to the theorem that we now know as Stokes' Theorem, when it was not he who invented the mathematical concept. Stokes was a distinguished professor of math and physics at Cambridge University where he made many scientific contributions to fluid dynamics, optics, and mathematical physics. Stokes first obtained knowledge of this theorem that related a surface integral to that of a line integral from William Thompson (Lord Kelvin) in a letter in 1850.[1] The theorem acquired its name from Stokes' habit of putting the theorem as a mathematical problem on the Cambridge prize examinations, resulting in its present name, Stokes' Theorem.

George Green, a self-taught English scientist, privately published "An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism" in 1828.[2] Only 100 copies were printed, which mostly went to his friends and family. Within this essay, a theorem equivalent to what we know as Green's theorem was documented, but was not widely known at the time of publication. Green entered Cambridge at the age of 40 to complete his undergraduate degree taking along with him his essay on electricity and magnetism. Only four years after graduating, Green died leaving behind his essay.

William Thompson (Lord Kelvin) also studied at Cambridge, and accidentally discovered a copy of Green's essay in 1846. He quickly realized the importance of what he had found and had the essay reprinted immediately. It was Lord Kelvin who popularized Green's work for future mathematicians, and made further advancements in math and science using Green's essay as a basis.[2]

II. STOKES' THEOREM

In order to understand Stokes' Theorem, one must first understand where it originated. Stokes' Theorem is a more complex version of Green's Theorem,[1] which states
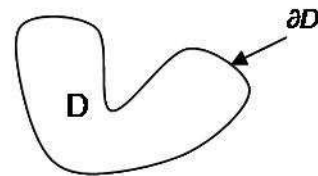

FIG. 1: A physical representation of the components of Greens Theorem.

the relationship between a line integral around a closed path and a double integral over the plane region bound by this path in $\mathbf{R}^2$ as shown Equation 1.

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \int \int_D (\nabla \mathrm{x} \mathbf{F}) d\mathbf{a} \tag{1}$$

D is the plane region and ∂D is the boundary of the closed path encompassing the plane region (Figure 1).

The left-hand side of the equation integrates the function, $\mathbf{F}$, with respect to the line enclosing the plane region evaluated over the boundary, ∂D. $\mathbf{F}$ is typically a vector field. The right-hand side of the equation has a double integral evaluating the curl of the vector field over the plane region, D. If a third dimension is added onto Green's Theorem, it now becomes Stokes' Theorem (Equation 2).

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_S (\nabla \mathrm{x} \mathbf{F}) d\mathbf{a} \tag{2}$$

S is the three-dimensional surface region that is bound by the closed path ∂S (Figure 2). The evaluation of the integrals in $\mathbf{R}^3$ follows the same form as Green's Theorem, but is slightly more complex since a third component has been added to the vector field. Stokes' Theorem states that the line integral around the boundary curve of S of the tangential component of $\mathbf{F}$ is equal to the surface integral of the normal component of the curl of $\mathbf{F}$. One can think of Green's Theorem as a special
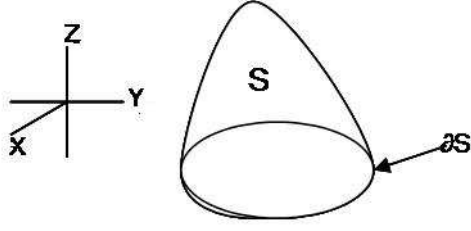
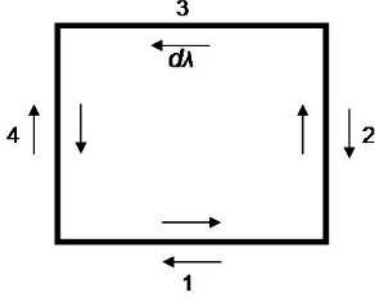FIG. 2: A physical representation of the components of Stokes Theorem.

FIG. 3: A rectangle showing the interior and exterior paths of the line integral.

case of Stokes' Theorem or vice versa since they are similarly related.

The orientation of the surface S will induce the positive orientation of ∂S. Moving along ∂S in a counterclockwise direction will yield the positive orientation of S, where as moving along ∂S in a clockwise direction will result in the negative orientation of S. Figure 2 assumes the positive orientation. Another way to check the orientation is to use the right hand rule with one's thumb pointing in the direction of the normal vector.

III.   PROOF OF STOKES' THEOREM

Looking at Stokes' Theorem in more detail, it can be broken down into a simple proof. From equation (2), Stokes' Theorem relates the surface integral of a derivative of a function and a line integral of that function with the path of integration being the perimeter bounding the surface[3]. If this surface is arbitrarily divided into many small rectangles, the circulation about one rectangle in the xy-plane can be observed (Figure 3). The circulation can be set up as scalar integrals as shown by equation(3).

$$
\begin{aligned}
circulation_{1234} &= \int_1 V_x(x,y)d\lambda_x + \int_2 V_y(x,y)d\lambda_y \quad (3)\\
&+ \int_3 V_x(x,y)d\lambda_x + \int_4 V_y(x,y)d\lambda_y
\end{aligned}
$$

Circulation$_{1234}$ follows the path around the rectangle as shown in figure (3). Each integral can be referred to the point (x_0, y_0) using a Taylor expansion to take into account the displacement of line segments 1 and 3 as well as segments 2 and 4. This results in equation(4).

$$
\begin{aligned}
circulation_{1234} &= \quad (4)\\
& V_x(x_0,y_0)dx + [V_y(x_0,y_0) + \frac{\partial V_y}{\partial x}dx]dy +\\
& [V_x(x_0,y_0) + \frac{\partial V_x}{\partial y}dy](-dx) + V_y(x_0,y_0)(-dy)\\
&= (\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y})dxdy
\end{aligned}
$$

If equation (4) is divided by $dxdy$, then the circulation per unit area is $\nabla \times \mathbf{V}_z$ which is given by the z-component of the vector. If this is applied to our one differential rectangle in the xy-plane, then equation (4) results in

$$
\sum_{all\ sides} \mathbf{V} \cdot d\lambda = \nabla \times \mathbf{V} \cdot d\sigma \quad (5)
$$

$d\lambda$ is the path taken around the interior and exterior of the rectangle, V is the vector being evaluated, and $d\sigma$ is the area of integration. If this is applied to all of the rectangles that make up the surface and using the definition of a Riemann integral, it is seen that the interior line segments of the rectangles will cancel leaving only the exterior line segments which make up the integral around the perimeter of the surface. Next taking the limit as the rectangles approach infinity with dx$\to$ 0 and dy$\to$ 0 results in equation 6

$$
\sum_{exterior\ line\ segments} \mathbf{V} \cdot d\lambda = \sum_{rectangles} \nabla \times \mathbf{V} \cdot d\sigma \quad (6)
$$

Writing equation 6 in integral form results in Stokes' Theorem.

$$
\oint \mathbf{V} \cdot d\lambda = \int_S \nabla \times \mathbf{V} \cdot d\sigma \quad (7)
$$

IV.   STOKES' THEOREM APPLICATIONS

Stokes' Theorem, sometimes called the Curl Theorem, is predominately applied in the subject of Electricity and Magnetism. It is found in the Maxwell-Faraday Law, and Ampere's Law.[4] In both cases, Stokes' Theorem is used to transition between the differential form and the integral form of the equation. In 1831 Michael Faraday conducted three experiments. One in which he pulled a loop of wire to the right through a magnetic field, one where he moved the magnet to the left holding the loop still, and one where both the loop of wire and the magnet were held

still with the strength of the magnetic field changing. The first two experiments resulted in motional emf, $\varepsilon = -\frac{d\phi}{dt}$, expressed by the flux rule. The last experiment resulted in the fact that a changing magnetic field induces an electric field as shown in equation 8.

$$\varepsilon = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi}{dt} \tag{8}$$

where E is the electric field, d$\mathbf{l}$ is the vector element that is part of the surface boundary, and $\frac{d\phi}{dt}$ is the change in flux with respect to time. E can be related to the change in B by replacing the change in flux with respect to time with the integral of the change in magnetic field with respect to time over a defined area. This is shown in equation 9

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \tag{9}$$

where $\mathbf{B}$ is the magnetic field, and d$\mathbf{a}$ is the vector element that is part of the surface, generally in 2 dimensions. Equation 9 is Faraday's law in integral form. To transform it into differential form, Stokes' Theorem can be used. Applying Stokes' Theorem to the left hand side of equation 9 yields

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int \int \nabla \times \mathbf{E} \cdot d\mathbf{a} \tag{10}$$

where E is the electric field, dl is a vector element that is part of the surface boundary, and d$\mathbf{a}$ is a vector element that is part of the surface.[5] Equation 10 carries sign ambiguity due to the assumption that the Right Hand Rule is used to find the direction of motion. As long as the integration of the surface does not vary with time, then we can differentiate equation 10 with respect to time under the integral sign resulting in equation 11

$$\frac{d}{dt} \int \int \mathbf{B} \cdot ds = \int \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \tag{11}$$

Since B is also a function of the coordinate system, then the partial derivative sign must be used. Combining equation 10 and equation 11 results in

$$\int \int \nabla \times \mathbf{E} \cdot d\mathbf{a} = \int \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \tag{12}$$

Because we assume that Faraday's Law must be true for every surface, it states that both of the vector integrals of equation 12 must be equal.[5] This transforms equation 12 into the Maxwell-Faraday equation in differential form (equation 13).

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{13}$$

The differential form of Faraday's law is one of Maxwell's equations which is why the equation is commonly referred to as the Maxwell-Faraday equation. The principal of the equation can be used as a basis for developing electric generators, inductors, and transformers.

Looking at Ampere's Law, it typically relates a magnetic field integrated around a closed loop to the electric current passing through the loop. If the electric field is constant throughout time, then Ampere's law relates the magnetic field ($\mathbf{B}$) to its source, the current density ($\mathbf{J}$) as shown in equation 14.

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} \tag{14}$$

where $\mathbf{B}$ is the magnetic field, dl is a vector element that is part of the surface boundary, μ_0 is the permeability of free space, and $\mathbf{J} \cdot$ d$\mathbf{a}$ is the total current passing through the 2 dimensional surface. This equation can be further reduced to

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} \tag{15}$$

where I_{enc} is the current enclosed by the surface boundary. This is Ampere's Law in integral form. To transform equation 15 into differential form, apply Stokes' Theorem to the left hand side of equation 15 and integrate with respect to time under the integral as shown for Maxwell-Faraday's equation. Doing so will result in equation 16.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{16}$$

Ampere's Law is useful for calculating the magnetic fields in highly symmetric cases when the magnitude of $\mathbf{B}$ can be taken out of the integral due to the fact that the magnitude of $\mathbf{B}$ is constant along the boundary. Some examples are calculating the magnetic field inside a long solenoid, inside a conductor, or from a long straight wire.

V. CONCLUSION

Although Sir George Gabriel Stokes did not invent Stokes' Theorem, it was named after him for his habit of putting the theorem on his tests at Cambridge University. When George Green entered Cambridge at the age of 40 to complete his undergraduate degree he brought with him his essay on electricity and magnetism which contained the original theorem that we know as Stokes's Theorem. Only four years after graduating, Green died leaving behind his essay at Cambridge where William Thompson discovered it and used it as a basis for further advancements in math and science. Stokes' Theorem states that the line integral around the boundary curve

of S of the tangential component of F is equal to the surface integral of the normal component of the curl of F. Stokes' Theorem can be applied to equations such as the Maxwell-Faraday Law and Ampere's Law to transition between the differential form and the integral form of the equation. Transitioning to the integral form of Ampere's Law allows for the calculation of the magnetic field inside solenoids, conductors, or from a long straight wire. The integral form of Maxwell-Faraday's Law allows for the calculation of an electric field from a changing magnetic field which is the basis for generators, inductors, and conductors. Thanks to the early works of Green and Thompson, Stokes' Theorem has contributed a great deal to the furthering of math and science.

[1] James Stewart. (1999) Calculus, 4th Edition, Brooks Cole Publishing, pg. $1102 - 1107$, $1139 - 1142$.

[2] Cambridge Encylopedia Vol 68, http://encyclopedia.stateuniversity.com/pages/20392/Sir-George-Gabriel-Stokes.html.

[3] George B. Arfken and Hans J. Weber. (2005) Mathematical Methods for Physicists, 6th Edition, Elsevier Academic Press, pg $64 - 67$.

[4] David J. Griffiths. (1999). Introduction to Electrodynamics, Third Edition, Upper Saddle River NJ: Prentice Hall, pp. $225 - 226$, $301 - 303$.

[5] Roger F. Harrington. (1958) Introduction to Electromagnetic Engineering, McGraw-Hill, New York, pg. $55 - 56$.

[6] Victor J. Katz. (1979) The History of Stokes' Theorem, Mathematics Magazine 52(3): 146-156.