# Bessel Functions and Their Applications 

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#### Abstract

Bessel functions are a series of solutions to a second order differential equation that arise in many diverse situations. This paper derives the Bessel functions through use of a series solution to a differential equation, develops the different kinds of Bessel functions, and explores the topic of zeroes. Finally, Bessel functions are found to be the solution to the Schroedinger equation in a situation with cylindrical symmetry.


## INTRODUCTION

While special types of what would later be known as Bessel functions were studied by Euler, Lagrange, and the Bernoullis, the Bessel functions were first used by F. W. Bessel to describe three body motion, with the Bessel functions appearing in the series expansion on planetary perturbation [1]. This paper presents the Bessel functions as arising from the solution of a differential equation; an equation which appears frequently in applications and solutions to physical situations [2] [3]. Frequently, the key to solving such problems is to recognize the form of this equation, thus allowing employment of the Bessel functions as solutions. The subject of Bessel Functions and applications is a very rich subject; nevertheless, due to space and time restrictions and in the interest of studying applications, the Bessel function shall be presented as a series solution to a second order differential equation, and then applied to a situation with cylindrical symmetry. Appropriate development of zeroes, modified Bessel functions, and the application of boundary conditions will be briefly discussed.

## THE BESSEL EQUATION

Bessel's equation is a second order differential equation of the form

$$
\begin{equation*}
x^{2} y^{\prime \prime}+x y^{\prime}+\left(x^{2}-\nu^{2}\right) y=0 \tag{1}
\end{equation*}
$$

By re-writing this equation as:

$$
\begin{equation*}
x\left(x y^{\prime}\right)^{\prime}+\left(x^{2}-\nu^{2}\right) y=0 \tag{2}
\end{equation*}
$$

and employing the use of a generalized power series, we re-write the terms of (2) in terms of the series:

$$
\begin{aligned}
y & =\sum_{n=0}^{\infty} a_{n} x^{n+s} \\
y^{\prime}= & \sum_{n=0}^{\infty} a_{n}(n+s) x^{n+s-1} \\
x y^{\prime}= & \sum_{n=0}^{\infty} a_{n}(n+s) x^{n+s}
\end{aligned}
$$

$$
\begin{aligned}
& \left(x y^{\prime}\right)^{\prime}=\sum_{n=0}^{\infty} a_{n}(n+s)^{2} x^{n+s-1} \\
& \left(x y^{\prime}\right)^{\prime}=\sum_{n=0}^{\infty} a_{n}(n+s)^{2} x^{n+s}
\end{aligned}
$$

When the coefficients of the powers of $x$ are organized, we find that the coefficient on $x^{s}$ gives the indicial equation $s^{2}-\nu^{2}=0, \Longrightarrow s= \pm \nu$, and we develop the general formula for the coefficient on the $x^{s+n}$ term:

$$
\begin{equation*}
a_{n}=-\frac{a_{n-2}}{(n+s)^{2}-\nu^{2}} \tag{3}
\end{equation*}
$$

In the case $s=\nu$ :

$$
\begin{equation*}
a_{n}=-\frac{a_{n-2}}{n(n+2 \nu)} \tag{4}
\end{equation*}
$$

and since $a_{1}=0, a_{n}=0$ for all $n=o d d$ integers. Coefficients for even powers of $n$ are found:

$$
\begin{equation*}
a_{2 n}=-\frac{a_{2 n-2}}{2^{2} n(n+\nu)} \tag{5}
\end{equation*}
$$

Recalling that for the gamma function:

$$
\begin{array}{r}
\Gamma(\nu+2)=(\nu+1) \Gamma(\nu+1), \\
\Gamma(\nu+3)=(\nu+2) \Gamma(\nu+2)=(\nu+2)(\nu+1) \Gamma(\nu+1)
\end{array}
$$

we can write the coefficients:

$$
\begin{array}{r}
a_{2}=-\frac{a_{0}}{2^{2}(1+\nu)}=-\frac{\Gamma(1+\nu)}{2^{2} \Gamma(2+\nu)} \\
a_{2 n}=-\frac{a_{0} \Gamma(1+\nu)}{n!2^{2 n} \Gamma(n+1+\nu)}
\end{array}
$$

Which allows us to write the terms of the series:

$$
\begin{equation*}
y=J_{\nu}(x)=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{\Gamma(n+1) \Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2 n+\nu} \tag{6}
\end{equation*}
$$

Where $J_{\nu}(x)$ is the Bessel function of the first kind, order $\nu$. The first five Bessel functions of this kind are shown in figure 1 .


FIG. 1: The Bessel Functions of orders $\nu=0$ to $\nu=5$

## DIFFERENT ORDERS OF BESSEL FUNCTIONS

In the preceding section, the form of Bessel functions were obtained are known as "Bessel functions of the first kind." [3]. Different kinds of Bessel functions are obtained with negative values of $\nu$, or with complex arguments. This section briefly explores these different kinds of functions

## Neumann Functions

Bessel functions of the second kind are known as Neumann functions, and are developed as a linear combination of Bessel functions of the first order described:

$$
\begin{equation*}
N_{\nu}(x)=\frac{\cos \nu \pi J_{\nu}(x)-J_{-\nu}(x)}{\sin \nu \pi} \tag{7}
\end{equation*}
$$

For integral values of $\nu$, the expression of $N_{\nu}(x)$ has an indeterminate form, and $\left.N_{\nu}(x)\right|_{x=0}= \pm \infty$. Nevertheless the limit of this function for $\mathrm{x} \neq 0$, the expression for $N_{\nu}$ is valid for any value of $\nu$, allowing the general solution to Bessel's equation to be written:

$$
\begin{equation*}
y=A J_{\nu}(x)+B N_{\nu}(x) \tag{8}
\end{equation*}
$$

with A and B as arbitrary constants determined from boundary conditions.

Bessel functions of the first and second kind are the most commonly found forms of the Bessel function in applications. Many applications in hydrodynamics, elasticity, and oscillatory systems have solutions that are based on the Bessel functions. One such example is that of a uniform density chain fixed at one end undergoing small oscillations. The differential equation of this situation is:

$$
\begin{equation*}
\frac{d^{2} u}{d z^{2}}+\frac{1}{z} \frac{d u}{d z}+\frac{k^{2} u}{z}=0 \tag{9}
\end{equation*}
$$

where $z$ references a point on the chain, $k^{2}=\frac{p^{2}}{g}$, with $p$ as the frequency of small oscillations at that point, and
$g$ the gravitational constant of acceleration. Eq. (9) is a form of eq. (1), and solution is:

$$
\begin{equation*}
u=A J_{0}\left(2 k z^{\frac{1}{2}}\right)+B Y_{0}\left(2 k z^{\frac{1}{2}}\right) \tag{10}
\end{equation*}
$$

where the A and B are determined by the boundary conditions.

## Modified Bessel Functions

Modified Bessel functions are found as solutions to the modified Bessel equation

$$
\begin{equation*}
x^{2} y^{\prime \prime}+x y^{\prime}-\left(x^{2}-\nu^{2}\right) y=0 \tag{11}
\end{equation*}
$$

which transforms into eq. (1) when $x$ is replaced with $i x$. However, this leaves the general solution of eq. (1) a complex function of $x$. To avoid dealing with complex solutions in practical applications [2], the solutions to (11) are expressed in the form:

$$
\begin{equation*}
I_{\nu}(x)=e^{\frac{\nu \pi i}{2}} J_{\nu}\left(x e^{\frac{i \pi}{2}}\right) \tag{12}
\end{equation*}
$$

The $I_{\nu}(x)$ are a set of functions known as the modified Bessel functions of the first kind. The general solution of the modified Bessel function is expressed as a combination of $I_{\nu}(x)$ and a function $I_{-\nu}(x)$ :

$$
\begin{equation*}
y=A I_{-\nu}(x)-B I_{\nu}(x) \tag{13}
\end{equation*}
$$

where again A and B are determined from the boundary conditions.

A solution for non-integer orders of $\nu$ is found:

$$
\begin{equation*}
K_{\nu}(x)=\frac{\pi}{2} \frac{I_{-\nu}(x)-I_{\nu}(x)}{\sin \nu \pi} \tag{14}
\end{equation*}
$$

The functions $K_{\nu}(x)$ are known as modified Bessel functions of the second kind. A plot of the Neumann Functions $\left(N_{\nu}(x)\right)$ and Modified Bessel functions $\left(I_{\nu}(x)\right)$ is shown in figure (2). A plot of the Modified Second Kind functions $\left(K_{n}(x)\right)$ is shown in fig. (3).

Modified Bessel functions appear less frequently in applications, but can be found in transmission line studies, non-uniform beams, and the statistical treatment of a relativistic gas in statistical mechanics.

## Zeroes of Bessel Functions

The zeroes of Bessel functions are of great importance in applications [5]. The zeroes, or roots, of the Bessel functions are the values of x where value of the Bessel function goes to zero $\left(J_{\nu}(x)=0\right)$. Frequently, the zeroes are found in tabulated formats, as they must the be numerically evaluated [5]. Bessel function's of the first


FIG. 2: The Neumann Functions (black) and the Modified Bessel Functions (blue) for integer orders $\nu=0$ to $\nu=5$


FIG. 3: The Modified Bessel Functions of the second kind for orders $\nu=0$ to $\nu=5$ [4]
and second kind have an infinite number of zeros as the value of $x$ goes to $\infty$. The zeroes of the functions can be seen in the crossing points of the graphs in figure (1), and figure (2). The modified Bessel functions of the first kind $\left(I_{\nu}(x)\right)$ have only one zero at the point $x=0$, and the modified Bessel equations of the second kind $\left(K_{\nu}(x)\right)$ functions do not have zeroes.

Bessel function zeros are exploited in frequency modulated (FM) radio transmission. FM transmission is mathematically represented by a harmonic distribution of a sine wave carrier modulated by a sine wave signal which can be represented with Bessel Functions. The carrier or sideband frequencies disappear when the modulation index (the peak frequency deviation divided by the modulation frequency) is equal to the zero crossing of the function for the $n^{t h}$ sideband.

## APPLICATION - SOLUTION TO SCHROEDINGER'S EQUATION IN A CYLINDRICAL WELL

Consider a particle of mass $m$ placed into a twodimensional potential well, where the potential is zero inside of the radius of the disk, infinite outside of the radius of the disk. In polar coordinates using r, $\phi$ as representatives of the system, the Laplacian is written:

$$
\begin{equation*}
\nabla^{2} \Psi=\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \Psi}{\partial r}\right)+\frac{1}{r^{2}} \frac{\partial^{2} \Psi}{\partial \phi^{2}} \tag{15}
\end{equation*}
$$

Which in the Schroedinger equation presents:

$$
\begin{equation*}
-\frac{\hbar^{2}}{2 m}\left[\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \Psi}{\partial r}\right)+\frac{1}{r^{2}} \frac{\partial^{2} \Psi}{\partial \phi^{2}}\right]=E \Psi \tag{16}
\end{equation*}
$$

Using the method of separation of variables with a proposed solution $\Psi=R(r) T(\phi)$ in (16), produces
$-\frac{\hbar^{2}}{2 m}\left[T(\phi) \frac{1}{R(r)} \frac{\partial}{\partial r}\left(r \frac{\partial R(r)}{\partial r}\right)+\frac{1}{r^{2}} R(r) \frac{\partial^{2} T}{\partial \phi^{2}}\right]=E R(r) T(\phi)$
and then dividing by $\Psi$ :

$$
\begin{equation*}
\left[\frac{1}{R} \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial R}{\partial r}\right)+\frac{1}{r^{2}} \frac{1}{T} \frac{\partial^{2} T}{\partial \phi^{2}}\right]=\frac{-2 m E}{\hbar^{2}} \tag{18}
\end{equation*}
$$

setting $\frac{2 m E}{\hbar^{2}}=k^{2}$ and multiplying through by $r^{2}$ produces

$$
\begin{equation*}
\frac{r}{R} \frac{d}{d r}\left(r \frac{d R}{d r}\right)+k^{2} r^{2}+\frac{1}{T} \frac{d^{2} T}{d \phi^{2}}=0 \tag{19}
\end{equation*}
$$

which is fully separated in $r$ and $\phi$. To solve, the $\phi$ dependent portion is set to $-m^{2}$, yielding the harmonic oscillator equation in $T(\phi)$, which presents the solution:

$$
\begin{equation*}
T(\phi)=A e^{i m \phi} \tag{20}
\end{equation*}
$$

Where A is a constant determined via proper normalization in $\phi$ :

$$
\begin{equation*}
\int_{0}^{2 \pi} A^{2} T(\phi) T(\phi) d \phi=1 \Longrightarrow A=\sqrt{\frac{1}{2 \pi}} \tag{21}
\end{equation*}
$$

Leaving the $\phi$ dependent portion $T(\phi)=\sqrt{\frac{1}{2 \pi}} e^{i m \phi}$
Working now with the $r$ dependent portion of the separated equation, multiplying the $r$ dependent portion of (19) by $r^{2}$, and setting equal to $m^{2}$ one obtains:

$$
\begin{equation*}
\frac{r^{2}}{R} \frac{d^{2} R}{d r^{2}}+\frac{r}{R} \frac{d R}{d r}+k^{2} r^{2}=m^{2} \tag{22}
\end{equation*}
$$

which when rearranged:

$$
\begin{equation*}
r^{2} \frac{d^{2} R}{d r^{2}}+r \frac{d R}{d r}+\left(k^{2} r^{2}-m^{2}\right)=0 \tag{23}
\end{equation*}
$$

which is of the same form as eq. (1), Bessel's differential equation. The general solution to eq. (23) is of the form of eq. (8), and we write that general solution:

$$
\begin{equation*}
R(r)=A J_{m}(k r)+B N_{m}(k r) \tag{24}
\end{equation*}
$$

where $J_{m}(k r)$ and $N_{m}(k r)$ are respectively the Bessel and Neumann functions of order $m$, and A and B are constants to be determined via application of the boundary conditions. As the solution must be finite at $x=0$, and as $N_{m}(k r) \rightarrow \infty$ as $x \rightarrow 0$, this means that the coefficient of $N_{m}(k r)=B=0$, leaving $R(r)$ to be expressed:

$$
\begin{equation*}
R(r)=A J_{m}(k r) \tag{25}
\end{equation*}
$$

Using the boundary condition that $\Psi=0$ at the radius of the disk, we have the condition that $J_{m}\left(k r_{b}\right)=0$, which implicitly requires the argument of $J_{m}$ to be a zero of the Bessel function. As noted earlier, these zeroes must be calculated individually in numerical fashion. Requiring that $k r_{b}=\alpha_{m, n}$, which is the $n t h$ zero of the $m^{t h}$ order Bessel function[6], the energy of the system is solved by expressing $k$ in terms of $\alpha_{m, n}$ in (eq. $k=\sqrt{\frac{2 m E}{\hbar^{2}}}$ ), arriving at:

$$
\begin{equation*}
E_{m, n}=\frac{\alpha_{m, n}^{2} \hbar^{2}}{2 m r_{b}^{2}} h \tag{26}
\end{equation*}
$$

The full solution for $\Psi$ is thus:

$$
\begin{equation*}
\Psi_{m}(r, \phi)=A J_{m}\left(\frac{\alpha_{m, n} r}{r_{b}}\right) e^{i m \phi} \tag{27}
\end{equation*}
$$

Because the Bessel function zeroes cannot be determined apriori, it is difficult to find a closed solution to express the normalization constant $A$. We select an order for $m$ to continue with the determination of the normalization constant, and arbitrarily choose $m=2$, which has a zero at $r=5.13562$, which we will set to be the radius of the circle. Given the preceding, the normalization can be for the $m=2, n=1$ case can be found:

$$
\begin{equation*}
\int_{0}^{r_{b o u n d a r y}} A^{2} J_{2}\left(\frac{\alpha_{2,1} r}{r_{b}}\right) J_{2}\left(\frac{\alpha_{2,1} r}{r_{b}}\right) d r=1 \tag{28}
\end{equation*}
$$

which for $r_{\text {boundary }}=5.13562 \Longrightarrow A=\sqrt{\frac{1}{0.510377}}$, (numerical values obtained using numerical integration in

Mathematica). Thus, we can express the full solution for the $m=2$ scenario:

$$
\begin{equation*}
\Psi(r, \phi)=\sqrt{\frac{1}{0.510377}} \sqrt{\frac{1}{2 \pi}} J_{2}\left(\frac{\alpha_{2,1} r}{r_{b}}\right) e^{i m \phi} \tag{29}
\end{equation*}
$$

And since we've effectively set $\alpha_{m, n}=r_{b}$,

$$
\begin{equation*}
\Psi(r, \phi)=\sqrt{\frac{1}{0.510377}} \sqrt{\frac{1}{2 \pi}} J_{2}(r) e^{i m \phi} \tag{30}
\end{equation*}
$$

Admittedly, this solution is somewhat contrived, but it shows the importance of working with the zeroes of the Bessel function to generate the particular solution using the boundary conditions.

## CONCLUSION

The Bessel functions appear in many diverse scenarios, particularly situations involving cylindrical symmetry. The most difficult aspect of working with the Bessel function is first determining that they can be applied through reduction of the system equation to Bessel's differential or modified equation, and then manipulating boundary conditions with appropriate application of zeroes, and the coefficient values on the argument of the Bessel function. This topic can be greatly expanded upon, and the reader is highly encouraged to review the applications and development presented in [2].

## REFERENCES

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