Stability Study of Buck Converter Using Eigenvalue theory

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(Dated:)

This paper introduces the eigenvalue theory and the application to linear dynamic systems to determine the stability of the system. This theory is used to analyze the stability of a buck converter.

INTRODUCTION

The general technique for the analysis and synthesis of linear systems is much more often used than that of nonlinear systems, the reasons for the predominance of linear method are, first, the general solutions for linear equations are more simply obtained than general solutions for a nonlinear equations; and second, many nonlinear equations can be adequately approximated by linear systems or linearize at an equilibrium point.

A linear dynamic system can be represented by an nth order differential equation. The nth order differential equation can be converted into a set of n first order differential equations expressed in terms of the state variables. For example, a circuit with two dynamic components: a capacitor C and an inductor L, and a resistor R. they connect in series. Take the current of the inductor and the voltage of the capacitor as the state variables, this dynamic system can be represented as

$$\begin{cases} \dfrac{di_L}{dt} = -\dfrac{1}{L}(v_c + i_L R) \\ \dfrac{dv_C}{dt} = \dfrac{i_L}{C} \end{cases} \tag{1}$$

Based on the linear system theory, several methods are proposed to determine the stability of a dynamic system. In frequency domain, which is the Laplace transformation of differential equations, Routh criteria, Nyquist's criteria are used.[1] In time domain, stability can be studies based on the solutions of the differential equations. Specifically, the Jacobian matrix of the state equations is used to determine the convergence of the solution, hence the stability of the system described by the state equations.[2]

Now we come to a specific series of dynamic systems, power electronics systems. Power electronics is the application of solid-state electronics for the control and conversion of electric power. Power electronic converters can be found wherever there is a need to modify the electrical energy form.[3] For example, the power supply for CPU of a computer is 5 volts DC, however, the power source which is accessible is 110V AC. Several power converters are needed to get the desired voltage. First, an AC-DC converter is required to convert the AC voltage to DC voltage, this converter is composed of a power electronics rectifier and a capacitor. Also feedback control is needed to keep the output voltage as a constant. Then, a step down converter is used to convert the 110V DC voltage to 48V DC voltage. Usually a Buck converter is used in the power stage and a controller is required to regulate the output voltage.[3] Finally, another step down converter is used to further decrease the voltage to 5 Volts.

In this paper, the stability for linear ordinary differential equations is firstly studied, then the mathematica background of eigenvalue theory is introduced. Then the stability of a Buck converter with a feedback controller is studied using the previously introduced theories.

MATHEMATICAL BACKGROUND REVIEW

Eigenvalue theory

Definition[2]:

(1)if F is a field and n is an positive integer, then $c \in F$ is an eigenvalue of a matrix $A \in M_{n\times n}(F)$, if and only if $|\lambda I - A| = 0$.

(2)A vector $v \in F^n \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is an eigenvector of A associated with λ, if and only if $Av = \lambda v$. Eigenvectors which associated with distinct eigenvalues are linearly independent.

Example:$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

then $det(A - \lambda I) = det(\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}) = 0$

$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0$

$\Rightarrow \lambda^2 - (a+d)\lambda + ad - bc = 0$

where $(a+d) = trace(A) = T$, $(ad-bc) = det(A) = D$, then the character equation can be rewrite as

$\lambda^2 - T\lambda + D = 0$.

solve the equation we get

$\lambda_1 = \frac{T-\sqrt{T^2-4D}}{2}, \lambda_2 = \frac{T+\sqrt{T^2-4D}}{2}$.

1) If $T^2 - 4D > 0$, both eigenvalues are real and distinct;

2) If $T^2 - 4D = 0$, $\lambda_1 = \lambda_2$;

3) If $T^2 - 4D > 0$, both eigenvalues are complex numbers, $\lambda_1 = \lambda_2^*$.

Substitute λ_i into $(\lambda_i I - A)v_i = 0$, v_i is the eigenvector associated with λ_i.

Proposition[2]:

If $det(A) \neq 0$, then A is diagonalizable. If eigenvalues of A are all real numbers, then $A = p\Lambda P^{-1}$, where Λ is a diagonal matrix composed of the eigenvalues of A, $\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$, n is the order of A. P is the matrix composed of eigenvectors.

Prove:

$P^{-1}AP$

$= P^{-1}A(v_1, v_2, \ldots, v(n))$

$= P^{-1}(Av_1, Av_2, \ldots, Av(n))$

$= P^{-1}(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v(n))$

$= P^{-1}\Lambda P$

$$= P^{-1}(\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix})P$$

$$= P^{-1}\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}P + P^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}P + \cdots + P^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}P = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix} = \Lambda$$

therefore, $PP^(-1)APP^(-1) = P\Lambda P^(-1)$

Stability of linear dynamic system

In this section we discuss the notion of stability. A fixed point is stable if the dynamical system can be forced to remain in any neighborhood of the fixed point by choice of initial data sufficiently close to that fixed point. It is asymptotically stable if, in addition, successive iterates starting near the fixed point approach it as n→ ∞. [4]

Example Consider the map $U_{n+1} = aU_n$ and the fixed point $\bar{U} = 0$. If $|a| \leq 1$ then the fixed point is stable: just choose $\delta = \varepsilon$ and then, since $|U_{n+1}| = |aU_n| \leq |U_n|$ we have $|U_n| \leq |U_0| = \varepsilon$ for all $n \geq 0$. If $|a| < 1$ then the fixed point is asymptotically stable since $U_n = a^n U_0$ and hence $U_n \to 0$ as $n \to \infty$.

Now let's consider the stability of a linear dynamic system. The linear dynamic system can be presented as a group of ordinary differential equation Eq. 2

$$\dot{u}(t) = Au(t) \tag{2}$$

The solution of Eq. 2 is simply

$$u(t) = e^{At}u(0).$$

[4]

Let A have eigenvalues $\lambda_i i = 1 \ldots n$ where n is the order of A. A can be rewrite as $A = P \bigwedge P^{-1}$, where $\bigwedge = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$ P is the matrix of eigenvectors. e^{At} can be rewrite as $Pe^{Jt}P^{-1}$. $e^{Jt} = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{\lambda_n t} \end{pmatrix}$

If and only if $Re(\lambda_i) < 0$ for all i, then $e^{\lambda_i} < 1$, when $t \to \infty$, $e^{\lambda_i t} \to 0$, then $e^{Jt} \to 0$, and $e^{At} = Pe^{Jt}P^{-1} \to 0$, there for $u(t) \to 0$, according to the definition of stability, $u(t)$ is stable at 0.

For any other linear dynamic system with forcing input, which the ODE is not homogeneous $\dot{u}(t) = Au(t) + C$, however, the stability of the solution is the same as $\dot{u}(t) = Au(t)$.

There for we can conclude that the linear dynamic system is stable if and only if the coefficient matrix of the correspond ODE has Re(λ_i)<0 for all i.

BUCK CONVERTER STABILITY ANALYSIS

Buck converter and the control circuit are shown in Figure 1. The operation principle can be illustrated using the equivalent circuits in Figure 2.

When the switch S_2 is closed, the equivalent circuit is as the left one in Figure 2. Call this State A. Take the inductor current i_{L_2}, output voltage v_0 and the control voltage v_{con}in control block as state variables, we can write the state equation during this time period.[5]

State A:

$$\begin{cases} \dfrac{di_{L_2}}{dt} = -\dfrac{v_0}{L_2} + \dfrac{E}{L_2} \\ \dfrac{dv_o}{d_t} = \dfrac{i_{L_2}}{C_2} - \dfrac{v_0}{\tau} \\ \dfrac{dv_{con}}{dt} = -K\dfrac{i_{L_2}}{C_2} + (\dfrac{1}{\tau} - \dfrac{1}{\tau_F})Kv_0 + K\dfrac{V_{ref}}{\tau_F}(1 + \dfrac{R_1}{R_2}) \end{cases} \tag{3}$$

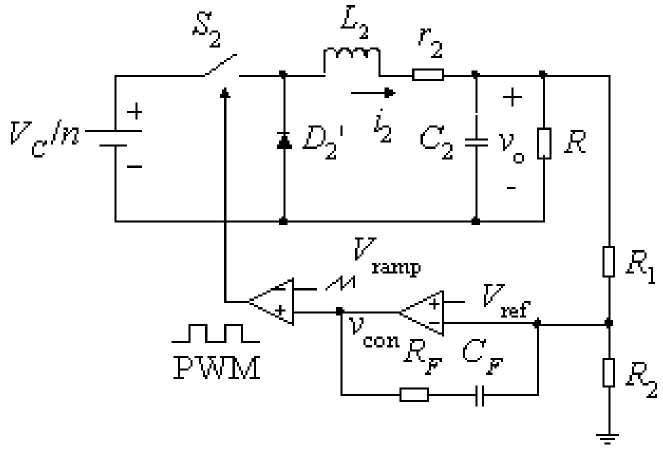

FIG. 1: Buck converter diagram.

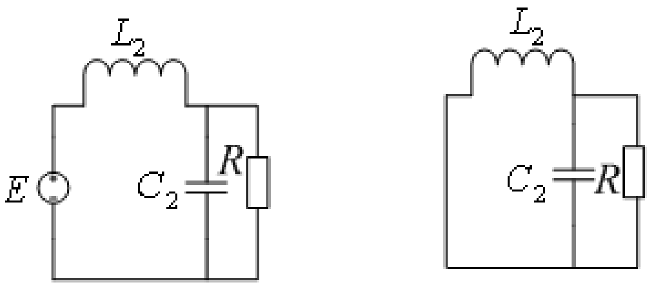

FIG. 2: Equivalent circuit.

Where $\tau = \sqrt{RC_2}$, $\tau_F = \sqrt{R_F C_F}$,K is called proportional coefficient, $K = \frac{R_F}{R_1}$.

When the switch is off, according to the equivalent circuit in the right one of Figure 2, we can get the state equations:

State B:

$$\begin{cases} \dfrac{di_{L_2}}{dt} = -\dfrac{v_0}{L_2} \\ \dfrac{dv_0}{dt} = \dfrac{i_{L_2}}{C_2} - \dfrac{v_0}{\tau} \\ \dfrac{dv_{con}}{dt} = -K\dfrac{i_{L_2}}{C_2} + (\dfrac{1}{\tau} - \dfrac{1}{\tau_F})Kv_0 + K\dfrac{V_{ref}}{\tau_F}(1 + \dfrac{R_1}{R_2}) \end{cases} \tag{4}$$

where d is duty ratio, which is $\frac{T_{on}}{T_{off}}$. In order to In steady state, the duty ratio d is a constant. It can be expressed as $D = \frac{V_{con} - V_L}{V_H - V_L}$, where V_H and V_L are the high and low boundary value of the triangular wave V_{camp}. Then the state equations can be averaged in one period.

TABLE I: Parameters in the circuit

parameters	values
E	60V
L_2	3mH
C_2	$47\mu F$
R	10Ω
V_{ref}	1.5V
R_F, C_F	$1.2k\Omega, 220nF$
T	$50\mu s$
V_H, V_L	8V,3V

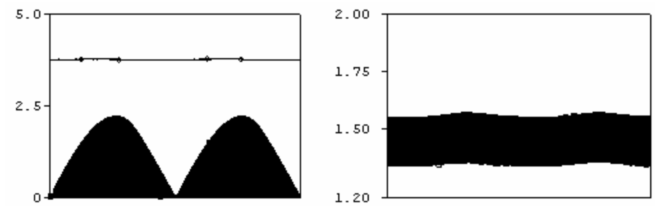

FIG. 3: System Stable.

averaged equation:

$$\begin{cases} \dfrac{\overline{di_{L_2}}}{dt} = -\dfrac{\overline{v_0}}{L_2} + D\dfrac{E}{L_2} \\ \dfrac{d\overline{v_o}}{d_t} = \dfrac{\overline{i_{L_2}}}{C_2} - \dfrac{\overline{v_0}}{\tau} \\ \dfrac{d\overline{v_{con}}}{dt} = -K\dfrac{\overline{i_{L_2}}}{C_2} + (\dfrac{1}{\tau} - \dfrac{1}{\tau_F})K\overline{v_0} + K\dfrac{V_{ref}}{\tau_F}(1 + \dfrac{R_1}{R_2}). \end{cases} \tag{5}$$

In steady state, the derivative of the state variables should be zero, therefore set the right hand side of Eq. 5 to be zero, we can get the equilibrium point Y_o:

$$Y_o = \begin{bmatrix} I_{L_2} \\ V_o \\ V_{con} \end{bmatrix} = \begin{bmatrix} \frac{DE}{R} \\ DE \\ V_L + D(V_H - V_L) \end{bmatrix}. \tag{6}$$

The coefficient matrix is

$$J(Y_o) = \begin{bmatrix} 0 & -\frac{1}{L_2} & \frac{E}{L(V_H - V_L)} \\ \frac{1}{C_2} & \frac{1}{\tau} & 0 \\ -\frac{K}{C_2} & K(\frac{1}{\tau} - \frac{1}{\tau_F}) & 0 \end{bmatrix}. \tag{7}$$

Then we use the calculated coefficient matrix to tell the stability of this buck converter. The parameters used in the circuit is listed in Table I. These parameters are decided according to the application of the circuit, while the proportional coefficient K is the one we should design, because it can affect the stability of the circuit. Now we calculate the eigenvalue of the Jacobian matrix Eq. 7 as K varies.

According to Table II we can see that when K is relatively small, all three real parts of the eigenvalues of Jacobian matrix are negative, therefore the circuit is stable.

TABLE II: Eigenvalue of coefficient matrix under different K

K	Eigenvalues	stability
0.014	$-39.0 \pm 3079.3i$, -985.8	stable
0.015	$-15.8 \pm 3115.1i, -1032.2$	stable
0.0156	$-4.6 \pm 3132.9i, -1054.6$	stable
0.0155	$-2.4 \pm 3136.5i, -1059.0$	stable
0.0157	$-2 \pm 3140.0i, -1063.4$	stable
0.0156	$6.3 \pm 3150.6i, -1076.4$	unstable

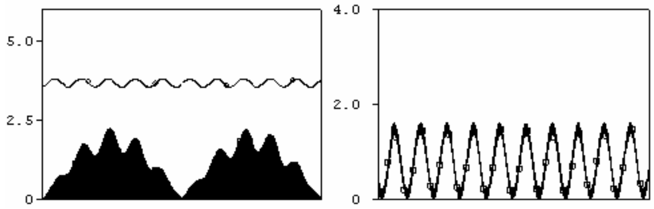

FIG. 4: System Unstable.

As K increase, the real part value also increase, when K increases to 0.0156, the real parts become positive, then the circuit loses stability. Figure 3 shows the simulation result of the voltage and current when the system is stable, Figure 4 shows the voltage and current when the system is unstable.

SUMMARY

This paper uses the eigenvalue theory to evaluate the stability of a linear dynamic system. Firstly, introduce some definition of eigenvalue theory; then use the theory to determine the stability of ordinary partial differential equations, which were used to describe linear dynamic system. Finally, use this theory in a buck converter to determine the stability.

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