Midterm Exam

P571

September 27, 2011

SHOW ALL WORK TO GET FULL CREDIT!

PART I:**ONLY TWO OF THE FOUR PROBLEMS WILL BE GRADED**. Take a look at the 4 problems. Each of them is worth 25 points. To make sure that you have enough time to do your work you will have to turn in only 2 of the 4 problems. **If you turn more than 2 problems only the two on top will be graded and 5 points will be deducted from your grade.**

PART II: Take the test home and bring **ALL** the problems solved on Tuesday October 4. Your grade for the test will be the **sum of the two** parts. A perfect score is worth 150 points.

Problem 1: The density of charge $\rho(\mathbf{x})$ is defined in such a way that

$$\int_V \rho(\mathbf{x}) d^3x = q,$$

where q is the total amount of charge inside the volume V.

Consider that a charge q is located at the point $(x, y, z) = (\sqrt{3}, -1, 2)$; consider that this is all the charge you have inside the volume V and consider that V is all the space, i.e., $-\infty \leq x_i \leq \infty$ in cartesian coordinates.

a) Provide an expression for $\rho(\mathbf{x})$ in cartesian coordinates.(7 points)

b) Provide an expression for $\rho(\mathbf{x})$ in spherical coordinates.(9 points)

c) Provide an expression for $\rho(\mathbf{x})$ in cylindrical coordinates.(9 points)

Problem 2: Consider the scalar field $\phi(x, y, z) = x^2 - yz$.

a) Find $\nabla\phi$ and evaluate it at the point $P = (3, -2, 1)$.(6 points)

b) Find a unit vector normal to the surface $\phi = 11$ at P.(6 points)

c) Find a unit vector normal to the surface $\phi = 11$ at $P' = (3, -1, 2)$.(6 points)

d) Calculate the angle between the vectors found in (b) and (c).(7 points)

Problem 3: An hexagonal Bravais lattice (in two dimensions) can be expanded by the lattice vectors:

$$\mathbf{a}_1 = a(1, 0),$$

and

$$\mathbf{a}_2 = a(\frac{1}{2}, \frac{\sqrt{3}}{2}),$$

where a is the lattice spacing. The vectors $\mathbf{a}_i$ are expressed in terms of their components (x_1, x_2) in the cartesian frame of reference S. We are going to define a frame of reference S' with oblique axis. The axis x'_1 is parallel to the vector $\mathbf{a}_1$ and the axis x'_2 is parallel to the vector $\mathbf{a}_2$.

a) What is the value of the angle α formed by the axis x'_1 and x'_2? (4 points)

b) Consider a vector

$$\mathbf{r} = a(\frac{5}{2}, \frac{\sqrt{3}}{2})$$

in S. Write $\mathbf{r}$ in terms of $\mathbf{a}_1$ and $\mathbf{a}_2$.(4 points)

c) Find r, the length of $\mathbf{r}$, in frame S. (4 points)

d) Write r'^i, i.e., the contravariant components of vector $\mathbf{r}$ in S' in terms of the cartesian components, i.e. provide $x'^i = x'^i(x_1, x_2)$.(4 points)

e) Write r'_i, i.e., the covariant components of vector $\mathbf{r}$ in S', i.e. provide $x'_i = x'_i(x_1, x_2)$.(4 points)

f) Find r', the length of $\mathbf{r}$, in system S'. Compare with your result in part (c) and comment.(5 points)

Problem 4:

Consider the tensor $\mathrm{G}^{\alpha\rho}$, where α and ρ take the values 0, 1, 2, and 3 with $x_0 = ct$ (c is a constant), $x_1 = x$, $x_2 = y$, and $x_3 = z$ in frame of reference S. In this frame

$$\mathrm{G}^{\alpha\rho} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -H_z & H_y \\ 0 & H_z & 0 & -H_x \\ 0 & -H_y & H_x & 0 \end{pmatrix}. \tag{1}$$

a) What is the rank of the tensor $\mathrm{G}^{\alpha\rho}$? Is $\mathrm{G}^{\alpha\rho}$ symmetric, antisymmetric or neither? (2.5 points)

b) Express $\mathrm{G}^{\alpha\rho}$ in system S' characterized by the variables $x'_0 = ct'$, $x'_1 = x'$, $x'_2 = y'$, and $x'_3 = z'$ knowing that the transformation matrix $M^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$ is given by

$$M^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where $\beta = v/c$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ where v and c are constants. Express the components of $G'^{\lambda,\delta}$ in terms of γ, β and the components of $G^{\alpha,\rho}$ given by Eq.(1). (5 points)

c) Find $(M^{-1})^{\mu}{}_{\nu}$, i.e., the inverse of $M^{\mu}{}_{\nu}$. (5 points)

d) Give a formal expression for the matrix elements of $(M^{-1})^{\mu}{}_{\nu}$ in terms of partial derivatives of the coordinates in S and S',i.e., the equivalent of the expression that I gave for $M^{\mu}{}_{\nu}$ in part (b).(5 points)

e) Now consider the tensor

$$\mathrm{G}^{\alpha}{}_{\rho} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -H_z & H_y \\ 0 & H_z & 0 & -H_x \\ 0 & -H_y & H_x & 0 \end{pmatrix}. \tag{3}$$

What is the rank of $\mathrm{G}^{\alpha}{}_{\rho}$? Is it a covariant, contravariant or mixed tensor? (2.5 points)

f) Provide an expression for $\mathrm{G}^{\alpha}{}_{\rho}$ in S'. Express the components of $G'^{\lambda}{}_{\delta}$ in terms of γ, β and the components of $G^{\alpha}{}_{\rho}$ given by Eq.(3). (5 points)