Midterm Exam

P571

September 27, 2011

SOLUTION:

**Problem 1**: We know that the charge $q$ is at $\mathbf{r}_0 = (x_0, y_0, z_0) = (\sqrt{3}, -1, 2)$.

a) In cartesian coordinates

$$\rho(\mathbf{x}) = q\delta(x - \sqrt{3})\delta(y + 1)\delta(z - 2)$$

since with the above definition

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \rho(\mathbf{x}) = q.$$

b) In spherical coordinates all we need to do is to express the $\delta$ function obtained in part (a) in terms of spherical coordinates. Then

$$\rho(\mathbf{x}) = \frac{q}{r^2 \sin\theta}\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)$$

where $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2} = 2\sqrt{2}$, $\cos\theta_0 = \frac{\sqrt{2}}{2}$ then $\theta_0 = \pi/4$, and $\tan\phi_0 = -1/\sqrt{3}$ then $\phi_0 = 11\pi/6$. Then

$$\rho(\mathbf{x}) = \frac{q}{r^2 \sin\theta}\delta(r - 2\sqrt{2})\delta(\theta - \pi/4)\delta(\phi - 11\pi/6),$$

since with the above definition

$$\int_{-0}^{\infty} r^2 dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \rho(\mathbf{x}) = q.$$

c) In cylindrical coordinates all we need to do is to express the $\delta$ function obtained in part (a) in terms of cylindrical coordinates $(\rho, \phi, z)$. Then

$$\rho(\mathbf{x}) = \frac{q}{\rho}\delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(z - z_0)$$

where $\rho_0 = \sqrt{x_0^2 + y_0^2} = 2$, $\tan\phi_0 = -1/\sqrt{3}$ then $\phi_0 = 11\pi/6$ and $z_0 = 2$. Then

$$\rho(\mathbf{x}) = \frac{q}{\rho}\delta(\rho - 2)\delta(\phi - 11\pi/6)\delta(z - 2),$$

since with the above definition

$$\int_{-0}^{\infty} \rho d\rho \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \rho(\mathbf{x}) = q.$$

**Problem 2**:

a)

$$\nabla\phi|_P = (2x, -z, -y)|_P = (6, -1, 2).$$

b) We see that $\phi|_P = 11$ then a unit normal vector to that surface at $P$ will be given by

$$\mathbf{V} = \frac{\nabla\phi|_P}{|\nabla\phi|_P|} = \frac{(6, -1, 2)}{\sqrt{41}}.$$

c) A unit vector perpendicular to the surface $\phi = 11$ at $P' = (3, -1, 2)$ is given by

$$\mathbf{V}' = \frac{\nabla\phi|'_P}{|\nabla\phi|'_P|} = \frac{(6,-2,1)}{\sqrt{41}}.$$

d) The angle between the two vectors can be found from

$$\mathbf{V}.\mathbf{V}' = VV'\cos\theta = \cos\theta,$$

since the two vectors are normalized. Then,

$$\cos\theta = \frac{36+2+2}{41} = \frac{40}{41}.$$

Thus

$$\theta = \cos^{-1}\theta = 12.68^o.$$

**Problem 3**:

a) Since the axis are parallel to the vectors $\mathbf{a}_i$ we can obtain the angle $\alpha$ between the axis from

$$\mathbf{a}_1.\mathbf{a}_2 = a_1 a_2 \cos\alpha.$$

Then

$$\cos\alpha = \frac{\mathbf{a}_1.\mathbf{a}_2}{a_1 a_2} = \frac{a^2}{a^2}\frac{1}{2} = \frac{1}{2}.$$

Then $\alpha = \pi/3 = 60^o$.

b) We know that

$$\mathbf{r} = a(\frac{5}{2}, \frac{\sqrt{3}}{2}) = \alpha\mathbf{a}_1 + \beta\mathbf{a}_2 = \alpha a(1,0) + \beta a(\frac{1}{2}, \frac{\sqrt{3}}{2}).$$

Then, comparing the $x$ and the $y$ components on both sides of the equation we can solve for $\alpha$ and $\beta$. We obtain $\alpha = 2$ and $\beta = 1$ then

$$\mathbf{r} = 2\mathbf{a}_1 + \mathbf{a}_2.$$

c) The length $r$ of vector $\mathbf{r}$ in frame S is given by

$$r = \sqrt{\mathbf{r}.\mathbf{r}} = \sqrt{a^2\frac{(25+3)}{4}} = \frac{a}{2}\sqrt{28} = a\sqrt{7}.$$

d) Since the contravariant components of $\mathbf{r}$ in S' are the components that arise from a projection parallel to the primed axis which are parallel to the $\mathbf{a}_i$ vectors we know that the components are the coefficients $\alpha$ and $\beta$ found in part (b) multiplied by $a$ which is the length of the $\mathbf{a}_i$ vectors. Then

$$r'^i = (2a, a) = a(2,1).$$

e) The covariant components are given by the projection normal to the primed axis. Then,

$$x'_1 = x^1 = a\frac{5}{2},$$

and

$$x'_2 = x^1\cos\alpha + x^2\sin\alpha = a\frac{5}{2}\frac{1}{2} + a\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} = a\frac{5}{4} + a\frac{3}{4} = 2a,$$

where we have used the value of $\alpha$ found in (a). Then

$$r'_i = (\frac{5}{2}a, 2a) = a(\frac{5}{2}, 2).$$

f) The length $r'$ of vector $\mathbf{r}$ in frame S' is given by

$$r' = \sqrt{r'_i r'^i} = \sqrt{(\frac{5}{2}a, 2a).(2a, a)} = \sqrt{5a^2 + 2a^2} = a\sqrt{7}.$$

As expected we obtain the same result as in part (c) because the length of the vector is an scalar and, thus, invariant under a change of reference frame.

**Problem 4**:

a) The rank of $G^{\alpha\rho}$ is 2. $G^{\alpha\rho}$ is an antisymmetric tensor because $G^{\alpha\rho} = -G^{\rho\alpha}$.
b) We know that since $G^{\alpha\rho}$ is a contravariant tensor

$$G'^{\lambda\delta} = \frac{\partial x'^\lambda}{\partial x^\alpha}\frac{\partial x'^\delta}{\partial x^\rho}G^{\alpha\rho} = M^\lambda{}_\alpha M^\delta{}_\rho G^{\alpha\rho}. \tag{1}$$

Notice that the only non-zero components of $G^{\alpha\rho}$ are $G^{12}$, $G^{13}$, $G^{21}$, $G^{23}$, $G^{31}$, and $G^{32}$. Then the only terms that are non-zero in Eq.(1) are:

$$G'^{\lambda\delta} = M^\lambda{}_1 M^\delta{}_2 G^{12}. \tag{2}$$

$$G'^{\lambda\delta} = M^\lambda{}_1 M^\delta{}_3 G^{13}. \tag{3}$$

$$G'^{\lambda\delta} = M^\lambda{}_2 M^\delta{}_1 G^{21}. \tag{4}$$

$$G'^{\lambda\delta} = M^\lambda{}_2 M^\delta{}_3 G^{23}. \tag{5}$$

$$G'^{\lambda\delta} = M^\lambda{}_3 M^\delta{}_1 G^{31}. \tag{6}$$

$$G'^{\lambda\delta} = M^\lambda{}_3 M^\delta{}_2 G^{32}. \tag{7}$$

The non-zero components of $M^\mu{}_\nu$ are $M^0{}_0$, $M^0{}_1$, $M^1{}_0$, $M^1{}_1$, $M^2{}_2$, and $M^3{}_3$. Then index $\lambda$ can take two possible values in Eqs.(2) and (3) and only one value in Eqs.(4), (5), (6), and (7) while index $\delta$ can take two possible values in Eqs.(4) and (6) and only one value in Eqs.(2), (3), (5), and (7). Then the non-zero values of $G'^{\lambda\delta}$ are:

$$G'^{02} = M^0{}_1 M^2{}_2 G^{12} = -\beta\gamma G^{12} = \beta\gamma H_z.$$

$$G'^{12} = M^1{}_1 M^2{}_2 G^{12} = \gamma G^{12} = -\gamma H_z.$$

$$G'^{03} = M^0{}_1 M^3{}_3 G^{13} = -\beta\gamma G^{13} = -\beta\gamma H_y.$$

$$G'^{13} = M^1{}_1 M^3{}_3 G^{13} = \gamma G^{13} = \gamma H_y.$$

$$G'^{20} = M^2{}_2 M^0{}_1 G^{21} = -\beta\gamma G^{21} = -\beta\gamma H_z.$$

$$G'^{21} = M^2{}_2 M^1{}_1 G^{21} = \gamma G^{21} = \gamma H_z.$$

$$G'^{23} = M^2{}_2 M^3{}_3 G^{23} = G^{23} = -H_x.$$

$$G'^{30} = M^3{}_3 M^0{}_1 G^{31} = -\beta\gamma G^{31} = \beta\gamma H_y.$$

$$G'^{31} = M^3{}_3 M^1{}_1 G^{31} = \gamma G^{31} = -\gamma H_y.$$

$$G'^{32} = M^3{}_3 M^2{}_2 G^{32} = G^{32} = H_x.$$

Then

$$G'^{\alpha\rho} = \begin{pmatrix} 0 & 0 & \beta\gamma H_z & -\beta\gamma H_y \\ 0 & 0 & -\gamma H_z & \gamma H_y \\ -\beta\gamma H_z & \gamma H_z & 0 & -H_x \\ \beta\gamma H_y & -\gamma H_y & H_x & 0 \end{pmatrix}.$$

c) Notice that $M^\mu{}_\nu$ is a block matrix, thus we only need to invert the $2 \times 2$ block formed by the components with $(\mu, \nu) = (0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. Using standard methods to invert the block we obtain that

$$(M^{-1})\mu_\nu = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

d) The expression requested is:

$$(M^{-1})^\mu{}_\nu = \frac{\partial x^\mu}{\partial x'^\nu}.$$

e) The rank of $G^\alpha{}_\rho$ is 2. It is a mixed tensor.
f) We proceed as in part (b). We know that since $G^\alpha{}_\rho$ is a mixed tensor

$$G'^\lambda{}_\delta = \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x^\rho}{\partial x'^\delta} G^\alpha{}_\rho = M^\lambda{}_\alpha (M^{-1})^\rho{}_\delta G^\alpha{}_\rho. \tag{8}$$

Notice that the only non-zero components of $G^\alpha{}_\rho$ are $G^1{}_2$,$G^1{}_3$,$G^2{}_1$,$G^2{}_3$,$G^3{}_1$, and $G^3{}_2$. Then the only terms that are non-zero in Eq.(1) are:

$$G'^\lambda{}_\delta = M^\lambda{}_1 (M^{-1})^2{}_\delta G^1{}_2. \tag{9}$$

$$G'^\lambda{}_\delta = M^\lambda{}_1 (M^{-1})^3{}_\delta G^1{}_3. \tag{10}$$

$$G'^\lambda{}_\delta = M^\lambda{}_2 (M^{-1})^1{}_\delta G^2{}_1. \tag{11}$$

$$G'^\lambda{}_\delta = M^\lambda{}_2 (M^{-1})^3{}_\delta G^2{}_3. \tag{12}$$

$$G'^\lambda{}_\delta = M^\lambda{}_3 (M^{-1})^1{}_\delta G^3{}_1. \tag{13}$$

$$G'^\lambda{}_\delta = M^\lambda{}_3 (M^{-1})^2{}_\delta G^3{}_2. \tag{14}$$

The non-zero components of $M^{\mu}{}_{\nu}$ are $M^0{}_0$, $M^0{}_1$, $M^1{}_0$, $M^1{}_1$, $M^2{}_2$, and $M^3{}_3$ and the non-zero components of $(M^{-1})^{\mu}{}_{\nu}$ are $(M^{-1})^0{}_0$, $(M^{-1})^0{}_1$, $(M^{-1})^1{}_0$, $(M^{-1})^1{}_1$, $(M^{-1})^2{}_2$, and $(M^{-1})^3{}_3$. Then index $\lambda$ can take two possible values in Eqs.(9) and (10) and only one value in Eqs.(11), (12), (13), and (14) while index $\delta$ can take two possible values in Eqs.(11) and (13) and only one value in Eqs.(9), (10), (12), and (14). Then the non-zero values of $G'^{\lambda}{}_{\delta}$ are:

$$G'^0{}_2 = M^0{}_1(M^{-1})^2{}_2 G^1{}_2 = -\beta\gamma G^1{}_2 = \beta\gamma H_z.$$

$$G'^1{}_2 = M^1{}_1(M^{-1})^2{}_2 G^1{}_2 = \gamma G^1{}_2 = -\gamma H_z.$$

$$G'^0{}_3 = M^0{}_1(M^{-1})^3{}_3 G^1{}_3 = -\beta\gamma G^1{}_3 = -\beta\gamma H_y.$$

$$G'^1{}_3 = M^1{}_1(M^{-1})^3{}_3 G^1{}_3 = \gamma G^1{}_3 = \gamma H_y.$$

$$G'^2{}_0 = M^2{}_2(M^{-1})^1{}_0 G^2{}_1 = \beta\gamma G^2{}_1 = \beta\gamma H_z.$$

$$G'^2{}_1 = M^2{}_2(M^{-1})^1{}_1 G^2{}_1 = \beta\gamma G^2{}_1 = \gamma H_z.$$

$$G'^2{}_3 = M^2{}_2(M^{-1})^3{}_3 G^2{}_3 = \beta\gamma G^2{}_3 = -\beta\gamma H_z.$$

$$G'^3{}_0 = M^3{}_3(M^{-1})^1{}_0 G^3{}_1 = \beta\gamma G^3{}_1 = -\beta\gamma H_z.$$

$$G'^3{}_1 = M^3{}_3(M^{-1})^1{}_1 G^3{}_1 = \beta\gamma G^3{}_1 = -\beta\gamma H_z.$$

$$G'^3{}_2 = M^3{}_3(M^{-1})^2{}_2 G^3{}_2 = \beta\gamma G^3{}_2 = -\beta\gamma H_z.$$

Then

$$G'^{\alpha}{}_{\rho} = \begin{pmatrix} 0 & 0 & \beta\gamma H_z & -\beta\gamma H_y \\ 0 & 0 & -\gamma H_z & \gamma H_y \\ \beta\gamma H_z & \gamma H_z & 0 & -H_x \\ -\beta\gamma H_y & -\gamma H_y & H_x & 0 \end{pmatrix}.$$