Midterm Exam #2

P571

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SOLUTION:

Problem 1:

a) The angle α between x'_1 and x'_2 is given by the angle between $\mathbf{r}$ and $\mathbf{x}_1$:

$$\cos\alpha = \frac{\mathbf{r}.\mathbf{e}_1}{|\mathbf{r}|} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \tag{1}$$

where $\mathbf{e}_1 = \frac{\mathbf{x}_1}{|\mathbf{x}_1|} = (1,0)$. Then

$$\alpha = \frac{\pi}{4} = 45^o. \tag{2}$$

b) We know that $\epsilon'_1 = (1,0)$, and $\epsilon'_2 = \frac{1}{\sqrt{2}}(1,1)$ then

$$g_{ij} = \vec{\epsilon}'_i.\vec{\epsilon}'_j = \begin{pmatrix} \vec{\epsilon}'_1.\vec{\epsilon}'_1 & \vec{\epsilon}'_1.\vec{\epsilon}'_2 \\ \vec{\epsilon}'_2.\vec{\epsilon}'_1 & \vec{\epsilon}'_2.\vec{\epsilon}'_2 \end{pmatrix} = \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 1 \end{pmatrix}. \tag{3}$$

c) The magnitude of r'^i is given by $(r'_i r'^i)^{1/2}$ then

$$r'_i r'^i = g_{ij} r'^j r'^i = g_{11} r'^1 r'^1 + g_{12} r'^2 r'^1 + g_{21} r'^1 r'^2 + g_{22} r'^2 r'^2 = 1 + \sqrt{2} + \sqrt{2} + 4 = 5 + 2\sqrt{2}, \tag{4}$$

and $|r'^i| = \sqrt{\sqrt{2} + \sqrt{2} + 4 = 5 + 2\sqrt{2}}$.

d)

$$T'^{ij} = r'^i r'^j = \begin{pmatrix} r'^1 r'^1 & r'^1 r'^2 \\ r'^2 r'^1 & r'^2 r'^2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}. \tag{5}$$

Then,

$$T'^i{}_j = g_{jk} T'^{ik} = \begin{pmatrix} g_{11} T'^{11} + g_{12} T'^{12} & g_{21} T'^{11} + g_{22} T'^{12} \\ g_{11} T'^{21} + g_{12} T'^{22} & g_{21} T'^{21} + g_{22} T'^{22} \end{pmatrix} = \begin{pmatrix} 1 + \sqrt{2} & 2 + \frac{\sqrt{2}}{2} \\ 2(1 + \sqrt{2}) & 4 + \sqrt{2} \end{pmatrix}. \tag{6}$$

Problem 2:

a)

$$[(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})]^i = \epsilon^{ijk} (\mathbf{A} \times \mathbf{B})_j (\mathbf{C} \times \mathbf{D})_k = \epsilon^{ijk} \epsilon_{jrs} A^r B^s \epsilon_{ktu} C^t D^u = \epsilon^{ijk} \epsilon_{ktu} \epsilon_{jrs} A^r B^s C^t D^u =$$

$$\epsilon^{kij} \epsilon_{ktu} \epsilon_{jrs} A^r B^s C^t D^u = (\delta^i{}_t \delta^j{}_u - \delta^i{}_u \delta^j{}_t) \epsilon_{jrs} A^r B^s C^t D^u = \epsilon_{jrs} A^r B^s C^i D^j - \epsilon_{jrs} A^r B^s C^j D^i =$$

$$A^r\epsilon_{rsj}B^sD^jC^i - A^r\epsilon_{rsj}B^sC^jD^i = A^r(\mathbf{B}\times\mathbf{D})_rC^i - A^r(\mathbf{B}\times\mathbf{C})_rD^i =$$

$$[\mathbf{A}.(\mathbf{B}\times\mathbf{D})]C^i - [\mathbf{A}.(\mathbf{B}\times\mathbf{C})]D^i. \tag{7}$$

Thus we see that the expression is a vector and Eq.(7) displays the component i of the vector. Then in vector, rather than tensor notation Eq.(7) becomes:

$$[(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})] = [\mathbf{A}.(\mathbf{B}\times\mathbf{D})]\mathbf{C} - [\mathbf{A}.(\mathbf{B}\times\mathbf{C})]\mathbf{D}. \tag{8}$$

b) In part (a) we saw that $[(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})]$ is a tensor of rank 1.

c) We see that $[(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})]$ is a pseudotensor because in Eq.(7) we found that

$$[(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})]^i = \epsilon^{ijk}\epsilon_{ktu}\epsilon_{jrs}A^rB^sC^tD^u = E^i. \tag{9}$$

We see that upon an inversion E^i will transform as a pseudovector i.e.

$$E'^i = (detI)\frac{\partial x'i}{\partial x^j}E^j, \tag{10}$$

where $(detI) = -1$ is the determinant of the inversion which is the way in which a pseudovector such as ϵ_{ijk} transforms upon an inversion. Since there is an odd number of Levi-Civita tensors in Eq.(9) upon an inversion we'll have $(detI)^3 = (detI) = -1$.

d) If the 4 vectors are coplanar $[(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})] = 0$. We can see this geometrically because the cross product of $\mathbf{A}$ and $\mathbf{B}$ will be a vector perpendicular to the plane to which the 4 vectors belong. The same will be the case for the cross product of $\mathbf{C}$ and $\mathbf{D}$. Then the cross product of the two parallel vectors will have to be zero.

In tensor notation it is also clear from Eq.(9), if we work in a system in which x_1 and x_2 define the plane in which the vectors are and x_3 is an axis perpendicular to that plane, then the indices j and k will have to label the direction perpendicular to the plane if $A^rB^sC^tD^u$ is non-zero. But this means that $j = k$ and thus $\epsilon^{ijk} = \epsilon^{ijj} = 0$.

Problem 3:

a) Since the boundary conditions are defined of flat surfaces we will work in cartesian coordinates. Using the solution to Laplace's equation in rectangular coordinates found in class and the boundary conditions we propose the following expression for the potential:

$$\Phi_V(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{a}e^{-\frac{n\pi x}{a}}, \tag{11}$$

where we have used that since $\Phi(x, y = 0) = 0$ the solution along y has to be of the form $\sin\alpha y$ and since $\Phi(x, y = a) = 0$, the $\alpha = \alpha_n = \frac{n\pi y}{a}$ with n ranging from 1 to ∞. Finally we have used that since $\Phi(x, y) = 0$ when $x \to \infty$ then along x the potential has to vanish exponentially as $e^{-\alpha_n x} = e^{-\frac{n\pi x}{a}}$.

Now we need to use the last boundary condition $\Phi(x = 0, y) = V$ to obtain A_n. Then at $x = 0$ Eq.(11) becomes

$$V = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{a}. \tag{12}$$

Now let's multiply both sides of the equation by $\sin\frac{m\pi y}{a}$ and integrate over y in the interval $(0, a)$. We obtain:

$$V\int_0^a \sin\frac{m\pi y}{a}dy = \sum_{n=1}^{\infty} A_n\int_0^a \sin\frac{m\pi y}{a}\sin\frac{n\pi y}{a}dy. \tag{13}$$

Eq.(13) becomes

$$-V\frac{a}{m\pi}\cos\frac{m\pi y}{a}|_0^a=\sum_{n=1}^{\infty}A_n\frac{a}{2}\delta_{mn}. \tag{14}$$

$$-V\frac{a}{m\pi}[(-1)^m-1]=A_m\frac{a}{2}. \tag{15}$$

Then

$$A_m=-V\frac{2}{m\pi}[(-1)^m-1] \tag{16}$$

We see that $A_m=0$ if m is even and

$$A_m=\frac{4V}{m\pi}, \tag{17}$$

if m is odd. Then the solution to the problem is

$$\Phi_V(x,y)=\frac{4V}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}\sin\frac{(2j+1)\pi y}{a}e^{-\frac{(2j+1)\pi x}{a}}. \tag{18}$$

b) Now let's consider the problem of a charge q located at $(a,a/2)$ inside the given volume with $\Phi=0$ on all the surfaces. Due to the presence of the charge that violates Laplace's equation, we will have to divide the space in two regions and propose two different solutions that will have to be matched at the plane surface $x=a$ that contains q. Thus we propose the following solutions:

For region I defined by $0\leq x\leq a$ and $0\leq y\leq a$:

$$\Phi_q^I(x,y)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi y}{a}sinh\frac{n\pi x}{a}, \tag{19}$$

where we have used that since $\Phi(x,y=0)=0$ the solution along y has to be of the form $\sin\alpha y$ and since $\Phi(x,y=a)=0$, the $\alpha=\alpha_n=\frac{n\pi y}{a}$ with n ranging from 1 to ∞ and that since $\Phi(x=0,y)=0$ the potential has to be proportional to $sinh\frac{n\pi x}{a}$. The constant A_n will be determined from the boundary conditions at $x=a$.

For region II defined by $x\geq a$ and $0\leq y\leq a$:

$$\Phi_q^{II}(x,y)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi y}{a}e^{-\frac{n\pi x}{a}}, \tag{20}$$

where we have used that since $\Phi(x,y=0)=0$ the solution along y has to be of the form $\sin\alpha y$ and since $\Phi(x,y=a)=0$, the $\alpha=\alpha_n=\frac{n\pi y}{a}$ with n ranging from 1 to ∞ and that since $\Phi(x,y)=0$ when $x\to\infty$ the potential has to vanish exponentially as $e^{-\alpha_n x}=e^{-\frac{n\pi x}{a}}$. The constant B_n will be determined from the boundary conditions at $x=a$.

At $x=a$ the potential has to be continuous then

$$\Phi_q^I|_{x=a}=\Phi_q^{II}|_{x=a}, \tag{21}$$

and also the normal component of the electric field $E_n=-\frac{\partial\Phi}{\partial n}=-\frac{\partial\Phi(x,y)}{\partial x}$, since the normal is along the x direction in this case, has a jump equal to σ/ϵ_0 where σ is the density of charge on the surface defined by $x=a$. In our case $\sigma=q\delta(y-a/2)$. Then we obtain the equation

$$-\frac{\partial\Phi_q^{II}(x,y)}{\partial x}|_{x=a}+\frac{\partial\Phi_q^I(x,y)}{\partial x}|_{x=a}=\frac{q}{\epsilon_0}\delta(y-\frac{a}{2}). \tag{22}$$

From Eq.(20) we obtain

$$A_n sinh n\pi=B_n e^{-n\pi}. \tag{23}$$

Then,

$$A_n = B_n \frac{e^{-n\pi}}{sinhn\pi}. \tag{24}$$

Using Eq.(24) in Eq.(22) we obtain:

$$\sum_{n=1}^{\infty} \frac{n\pi}{a} B_n \sin\frac{n\pi y}{a} e^{-n\pi} + \sum_{n=1}^{\infty} \frac{n\pi}{a} B_n \frac{e^{-n\pi}}{sinhn\pi} \sin\frac{n\pi y}{a} coshn\pi = \frac{q}{\epsilon_0}\delta(y-\frac{a}{2}). \tag{25}$$

$$\sum_{n=1}^{\infty} \frac{n\pi}{a} B_n \sin\frac{n\pi y}{a} e^{-n\pi}(1+cotanhn\pi) = \frac{q}{\epsilon_0}\delta(y-\frac{a}{2}). \tag{26}$$

Now let's multiply both sides of the equation by $\sin\frac{m\pi y}{a}$ and integrate over y in the interval $(0,a)$. We obtain:

$$\frac{m\pi}{2} B_m e^{-m\pi}(1+cotanhm\pi) = \frac{q}{\epsilon_0}\sin\frac{m\pi}{2}. \tag{27}$$

Since $e^{-m\pi}(1+cotanhm\pi) = 1/sinhn\pi$ and $\sin\frac{m\pi}{2}$ is non-zero only if m is odd we find that $B_m = 0$ for m even and for m odd

$$B_m = \frac{2q}{m\pi\epsilon_0}\sin\frac{m\pi}{2}\sinh m\pi, \tag{28}$$

or relabeling $m = 2j+1$

$$B_{2j+1} = \frac{2q}{(2j+1)\pi\epsilon_0}(-1)^j \sinh(2j+1)\pi. \tag{29}$$

Then, replacing Eq.(29) in Eq.(24) we find that

$$A_{2j+1} = \frac{2q}{(2j+1)\pi\epsilon_0}(-1)^j e^{-(2j+1)\pi}. \tag{30}$$

Then replacing Eq.(30) in Eq.(19) and Eq.(29) in Eq.(20) we obtain that

$$\Phi_q^I(x,y) = \frac{2q}{\pi\epsilon_0}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1} e^{-(2j+1)\pi}\sin\frac{(2j+1)\pi y}{a} sinh\frac{(2j+1)\pi x}{a}, \tag{31}$$

and

$$\Phi_q^{II}(x,y) = \frac{2q}{\pi\epsilon_0}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\sinh(2j+1)\pi\sin\frac{(2j+1)\pi y}{a} e^{-\frac{(2j+1)\pi x}{a}}. \tag{31}$$

Both equations can be written as

$$\Phi_q(x,y) = \frac{2q}{\pi\epsilon_0}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\sin\frac{(2j+1)\pi y}{a}\sinh\frac{(2j+1)\pi x_<}{a} e^{-\frac{(2j+1)\pi x_>}{a}}, \tag{32}$$

where $x_<$ ($x_>$) is the smaller (larger) between x and a.

c) To solve this problem we can use the superposition principle combining the solutions to parts (a) and (b) so that

$$\Phi(x,y) = \Phi_V(x,y) + \Phi_q(x,y). \tag{33}$$

Problem 4:

a) Since the covariant metric tensor is used to lower indices and we need to lower two indices we obtain

$$G_{\mu\nu} = g_{\mu\alpha} g_{\nu\rho} G^{\alpha\rho}. \tag{34}$$

b) The tensor $G_\mu{}^\mu$ has rank 0 since there are no free indices. It should be zero because it is the trace of the tensor G which is traceless, but we can also see this from the explicit expression:

$$G_\mu{}^\mu = g_{\mu\alpha} G^{\alpha\mu} = g_{00} G^{00} + g_{11} G^{11} + g_{22} G^{22} + g_{33} G^{33} = 0, \tag{35}$$

where we have used that g_{ij} is diagonal .

c) The tensor $G_{\mu\nu} G^{\alpha\rho}$ has rank 4. For the element requested we have:

$$G_{13} G^{12} = g_{1\alpha} g_{3\beta} G^{\alpha\beta} G^{12} = g_{11} g_{33} G^{13} G^{12} = (-1)(-1) H_y (-H_z) = -H_y H_z, \tag{36}$$

where we have used that since g_{ij} is diagonal we only have contributions different from zero if $\alpha = 1$ and $\beta = 3$.

d) The tensor $G_{\mu\nu} G^{\nu\rho}$ has rank 2 because only two of the four indices are not contracted. For the element requested we have:

$$G_{1\nu} G^{\nu 2} = g_{1\alpha} g_{\nu\beta} G^{\alpha\beta} G^{\nu 2} = g_{11} g_{\nu\beta} G^{1\beta} G^{\nu 2} = g_{11} g_{33} G^{13} G^{32} = (-1)(-1) H_y H_x = H_y H_x, \tag{37}$$

where we have used that given the form of G the only values of ν that will give us a non-zero element of the form $G^{\nu 2}$ are $\nu = 1$ or 3 and the only values of β that will give us a non-zero element of the form $G^{1\beta}$ are $\beta = 2$ or 3. Then, since $g_{\nu\beta}$ is diagonal we need $\nu = \beta$ which means that $\nu = \beta = 3$ provides the only non-zero contrinution.