Homework #11

Problem 4:

i) Green Function approach:

Since the potential on the surfaces is given we need to use the Green function for Dirichlet boundary conditions that was obtained in class:

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{a}{r'|\mathbf{r}-\frac{a^2\hat{\mathbf{n}}}{r'}|}. \tag{1}$$

The next step is to perform its expansion in terms of spherical harmonics using the expression obtained in class:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r_<^l}{r_>^{l+1}}\frac{Y_l^m(\theta,\phi)Y_l^{-m}(\theta',\phi')}{(2l+1)}. \tag{2}$$

where $r_<$ ($r_>$) is the smaller (larger) of r' and r. The expansion of the second term has a similar form but we know that in this problem, in which the volume considered is the volume outside the sphere of radius a, $r > a^2/r'$ because $r > a$ always, and $r' > a$ always which means that $a/r' < 1$ and thus $a^2/r' < a$. Then

$$\frac{1}{|\mathbf{r}-\frac{a^2\hat{\mathbf{n}}}{r'}|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(a^2/r')^l}{(r)^{l+1}}\frac{Y_l^m(\theta,\phi)Y_l^{-m}(\theta',\phi')}{(2l+1)} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{a^{2l}}{r'^l(r)^{l+1}}\frac{Y_l^m(\theta,\phi)Y_l^{-m}(\theta',\phi')}{(2l+1)}. \tag{3}$$

Replacing (2) and (3) in (1) we obtain:

$$G(\mathbf{r},\mathbf{r}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{(2l+1)}[\frac{r_<^l}{r_>^{l+1}} - \frac{a^{2l}}{(rr')^{l+1}}]Y_l^m(\theta,\phi)Y_l^{-m}(\theta',\phi'). \tag{4}$$

where $r_<$ ($r_>$) is the smaller (larger) between r and r'.

In the problem we are considering the density of charge is zero thus, only the surface integral contributes to the potential outside the sphere which is given by

$$\Phi(\mathbf{r}) = \frac{-1}{4\pi}\oint_S \Phi_s\frac{\partial G}{\partial n'}dS'. \tag{5}$$

In this case $n' = -r'$ since we need to consider the normal that points outside the volume which is the exterior of the sphere of radius a. Also notice that r' has to be on the surface, i.e., $r' = a$ while r is outside the sphere, then $r' < r$ which means that in Eq.(4) we have to use $r_< = r'$ and $r_> = r$. Then we obtain

$$-\frac{\partial G}{\partial r'}|_{r'=a} = -4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{(2l+1)}[\frac{la_<^{l-1}}{r^{l+1}} + (l+1)\frac{a^{2l+1}}{a^{l+2}r^{l+1}}]Y_l^m(\theta,\phi)Y_l^{-m}(\theta',\phi'). \tag{6}$$

Plugging Eq.(6) in Eq.(5) we obtain:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{(2l+1)}Y_l^m(\theta,\phi)a^2[\frac{la_<^{l-1}}{r^{l+1}} + \frac{(l+1)a^{2l+1}}{a^{l+2}r^{l+1}}]\int_{-1}^{1}d(\cos\theta')\Phi_S\int_0^{2\pi}Y_l^{-m}(\theta',\phi'). \tag{7}$$

We see that

$$\int_0^{2\pi}Y_l^{-m}(\theta',\phi') = 2\pi\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta')\delta_{m,0}. \tag{8}$$

Inserting Eq.(8) in Eq.(7) we obtain:

$$\Phi(r,\theta) = \frac{1}{2}\sum_{l=0}^{\infty} P_l(\cos\theta)(2l+1)\frac{a^{l+1}}{r^{l+1}}[-V_0\int_{-1}^{0} d(\cos\theta')P_l(\cos\theta') + V_0\int_{0}^{1} d(\cos\theta')P_l(\cos\theta')]. \quad (9)$$

Which using the symmetry properties of the Legendre polynomials becomes

$$\Phi(r,\theta) = \sum_{j=0}^{\infty}(4j+3)P_{2j+1}(\cos\theta)(\frac{a}{r})^{2j+2}V_0\int_{0}^{1} d(\cos\theta')P_{2j+1}(\cos\theta'). \quad (10)$$

Using 12.3.8: $\int_0^1 P_{2j+1}(x)dx = \frac{P_{2j}(0)}{2j+1} = \frac{(-1)^j(2j-1)!!}{2j+2!!}$,

$$\Phi(r,\theta) = V_0\sum_{j=0}^{\infty}(4j+3)P_{2j+1}(\cos\theta)\frac{(-1)^j(2j-1)!!}{2j+2!!}(\frac{a}{r})^{2j+2}. \quad (11)$$

ii) Separation of variables:

Notice that we only care about the region given by $r \geq a$, there are no free charges, the b.c. are given on a spherical surface, and the problem has azimuthal symmetry; then we propose a solution to Laplace's equation in spherical coordinates and independent of ϕ:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\frac{A_l}{r^{l+1}}P_l(\cos\theta), \quad (12)$$

where we have used the fact that the potential vanishes as $r \to \infty$ and then we cannot have positive powers of r in the potential. In order to obtain the coefficients A_l we will use the boundary condition that $\Phi(r=a,\theta) = V_S$ with $V_S = V_0$ for $0 \leq \theta \leq \pi/2$ and $V_S = -V_0$ for $\pi/2 < \theta \leq \pi$:

$$\Phi(r=a,\theta) = \sum_{l=0}^{\infty}\frac{A_l}{a^{l+1}}P_l(\cos\theta) = V_S, \quad (13)$$

Now we multiply both sides of Eq.(13) by $P_n(\cos\theta)$ and integrate over $\cos\theta$ ranging from -1 to 1 taking advantage of the orthogonality properties of the Legendre polynomials:

$$\sum_{l=0}^{\infty}\frac{A_l}{a^{l+1}}\int_{-1}^{1} d(\cos\theta)P_l(\cos\theta)P_n(\cos\theta) = V_0(-\int_{-1}^{0} d(\cos\theta)P_n(\cos\theta) + \int_{0}^{1} d(\cos\theta)P_n(\cos\theta)). \quad (14)$$

We know that

$$\int_{-1}^{1} d(\cos\theta)P_l(\cos\theta)P_n(\cos\theta) = \frac{2}{2l+1}\delta_{l,n}, \quad (15)$$

and

$$(-\int_{-1}^{0} d(\cos\theta)P_n(\cos\theta) + \int_{0}^{1} d(\cos\theta)P_n(\cos\theta)) = (\int_{0}^{1} d(-\cos\theta)P_n(\cos\theta) + \int_{0}^{1} d(\cos\theta)P_n(\cos\theta)). \quad (16)$$

Eq.(16) vanishes if n is even while if it is odd, i.e., $n = 2j+1$, the integral equals $2\int_0^1 d(\cos\theta)P_{2j+1}(\cos\theta)) = 2\frac{P_{2j}(0)}{2j+1} = 2\frac{(-1)^j(2j-1)!!}{2j+2!!}$ as discussed in part (i) of the problem,then

$$\frac{A_n}{a^{n+1}}\frac{2}{2n+1} = 0 \quad (17)$$

for n even so that $A_n = 0$ for n even and for odd $n = 2j+1$ we obtain

$$\frac{A_{2j+1}}{a^{2j+2}} \frac{2}{4j+3} = V_0 2 \frac{(-1)^j (2j-1)!!}{2j+2!!}. \tag{18}$$

Then

$$A_{2j+1} = V_0 \frac{(-1)^j (2j-1)!!}{2j+2!!} a^{2j+2} (4j+3). \tag{19}$$

Plugging Eq.(19) into Eq.(12) we obtain

$$\Phi(r,\theta) = V_0 \sum_{j=0}^{\infty} (4j+3) P_{2j+1}(\cos\theta) \frac{(-1)^j (2j-1)!!}{2j+2!!} (\frac{a}{r})^{2j+2}, \tag{20}$$

which, as expected, is the same result as Eq.(11).