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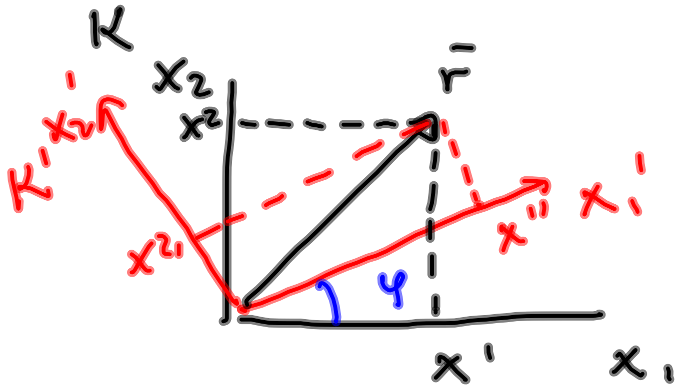
Last time:



we found that

$$① \begin{cases} x_1' = x_1 \cos\varphi + x_2 \sin\varphi \\ x_2' = -x_1 \sin\varphi + x_2 \cos\varphi \end{cases}$$

① Can be written as:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}}_{M} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

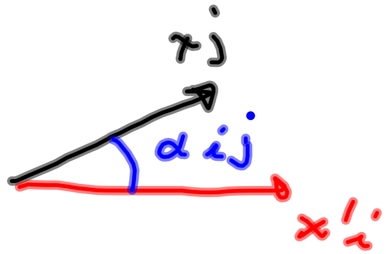
or $\quad x_i' = \sum_{j=1}^{2} M_{ij} x_j \equiv M_{ij} x_j$

Einstein's notation. You sum over repeated indices.

What are the elements of M in terms of x_i and x'_i?

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1'}{\partial x_1}, & \frac{\partial x_1'}{\partial x_2}, \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} \end{pmatrix}$$

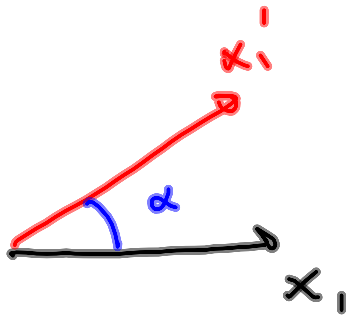
$$x_1' = \cos\varphi\, x_1 + \sin\varphi\, x_2$$
$$x_2' = -\sin\varphi\, x_1 + \cos\varphi\, x_2$$



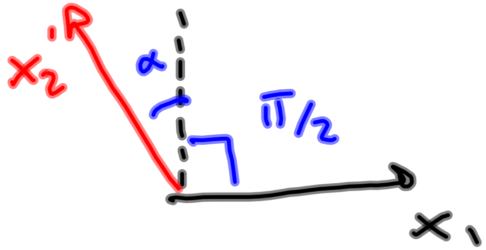
Then $m_{ij} = \dfrac{\partial x_i'}{\partial x_j}$

Notice that for a rotation $\dfrac{\partial x_i'}{\partial x_j} = \cos\alpha_{ij}$

Ex:



$$\frac{\partial x_1'}{\partial x_1} = \cos\alpha = \cos\alpha_{11}$$



$$\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin\alpha = \frac{\partial x_2'}{\partial x_1} = \cos\alpha_{21}$$

Generalizing to dimension N:

A contravariant vector $x^i = (x_1, x_2, \dots x_N)$

transforms from K to K' as

$$\boxed{x^{i'} = \frac{\partial x^{i'}}{\partial x^j} x^j}$$ sum over j.

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Example: transformation of a scalar.

$N=2$

In K $\quad |\bar{r}| = (x_1^2 + x_2^2)^{1/2}$

In K' $\quad |\bar{r}'| = (x_1'^2 + x_2'^2)^{1/2} =$

$$= \left[(x_1 \cos\varphi + x_2 \sin\varphi)^2 + (-x_1 \sin\varphi + x_2 \cos\varphi)^2\right]^{1/2}$$

$$= (x_1^2 + x_2^2)^{1/2} = |\bar{r}| \quad \text{invariant.}$$

Length is an invariant

$$|x_i| = \left[\sum_i x_i x_i\right]^{1/2} \doteq (x_i x^i)^{1/2} = \left[(x_1, x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right]^{1/2} = (x_1^2 + x_2^2)^{1/2}$$

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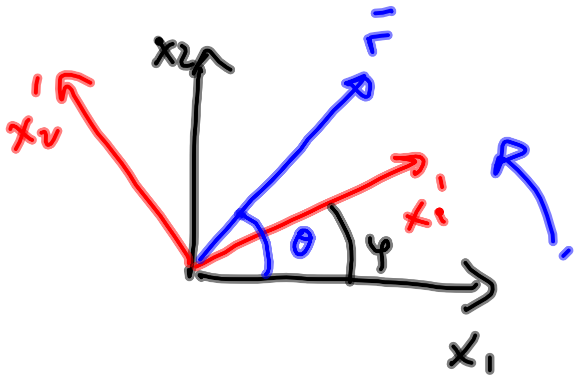
Then $\bar{r} = x^i$ is our prototype contravariant vector.

It transforms as $x^{i'} = \dfrac{\partial x^{i'}}{\partial x^j} x^j$ and

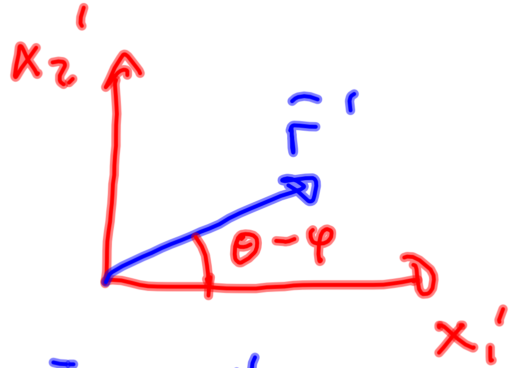
it is represented as a column vector.

Why contravariant ?



axis rotate counterclockwise

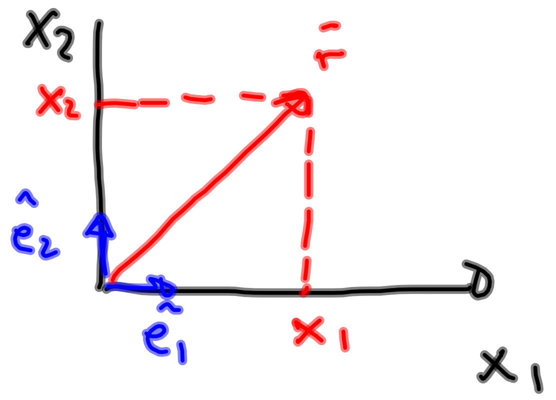
In K' system:



$\bar{r}$ in K' seems to have rotated clockwise.

Covariant vectors

Consider in $\mathbb{R}$



$\hat{e}_i$: is a vector of unit length along x_i.

$$\bar{r} = x_1 \hat{e}_1 + x_2 \hat{e}_2 = \sum_{i=1}^{2} x^i \hat{e}_i$$

as matrices if x^i is a column vector then $\hat{e}_i$ have to be row vectors:

$$(\hat{e}_1, \hat{e}_2) \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = x^1 \hat{e}_1 + x^2 \hat{e}_2$$

Vectors expressed as row vectors are call covariant.

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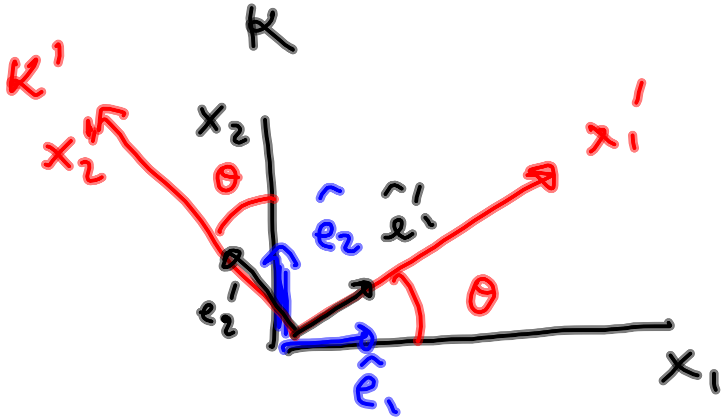
Let's see how the base vectors transform under a rotation:



$$\hat{e}_1' = \cos\theta\, \hat{e}_1 + \sin\theta\, \hat{e}_2$$

$$\hat{e}_2' = -\sin\theta\, \hat{e}_1 + \cos\theta\, \hat{e}_2$$

$$(\hat{e}_1', \hat{e}_2') = (\hat{e}_1, \hat{e}_2) \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{A} = \sum_{j=1}^{N} \hat{e}_j A_{ji} \equiv$$

$$\equiv \hat{e}_j A_j{}^i \quad \text{Einstein notation.}$$

Notice that $A = M^T$

and $AM = 1$ because $M^T = M^{-1}$ for an orthogonal matrix.

$M^T M = M^{-1} M = \mathbb{I}$

Now let's see the form of the elements of A in terms of x^i and x'^j:

$$M^i{}_j = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} \\ \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} \end{pmatrix} \equiv \frac{\partial x'^i}{\partial x^j}$$

$$A^i{}_j = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial x^{1'}} & \frac{\partial x^1}{\partial x^{2'}} \\ \frac{\partial x^2}{\partial x^{1'}} & \frac{\partial x^2}{\partial x^{2'}} \end{pmatrix} \equiv \frac{\partial x^i}{\partial x'^j}$$

$$x^1 = x^{1'}\cos\theta \cdot x^{2'}\sin\theta$$

$$x^2 = x^{1'}\sin\theta - x^{2'}\cos\theta$$

Then a covariant vector transforms from K to K' as:

$$\boxed{x'_i = \frac{\partial x^i}{\partial x'^j}\, x_j}$$

covariant

Then we found that:

$$x^{i'} = \frac{\partial x'^i}{\partial x^j}\, x^j$$

contravariant

The index up reminds us that x' is up in the matrix elements.

$$\hat{e}'_j = \frac{\partial x^i}{\partial x'^j}\, \hat{e}_j$$

covariant

The index down reminds us that the x' variable is down in the matrix elements.

Covariant and contravariant components.

A contravariant (or covariant vector) can be expressed in terms of contravariant or covariant components: We need to do this to calculated, for example, scalar products of vectors.

Ex:

$$|\bar{r}|^2 = \bar{r}\cdot\bar{r} = \sum_{i=1}^{N} x_i x_i$$

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How do we write it in terms of matrices?

Define: $(x_1, x_2, \ldots, x_N)$ as the covariant form

of $\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^N \end{pmatrix}$ (contravariant)

Now:

$$|\bar{r}|^2 = x_i x^i = (x_1, x_2, \ldots, x_N)\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^N \end{pmatrix} = x_1^2 + x_2^2 + \ldots + x_N^2 = \bar{r}\cdot\bar{r}$$

In cartesian coordinates

$x_i = x^i$ (component by component)

but NOT true in other systems.

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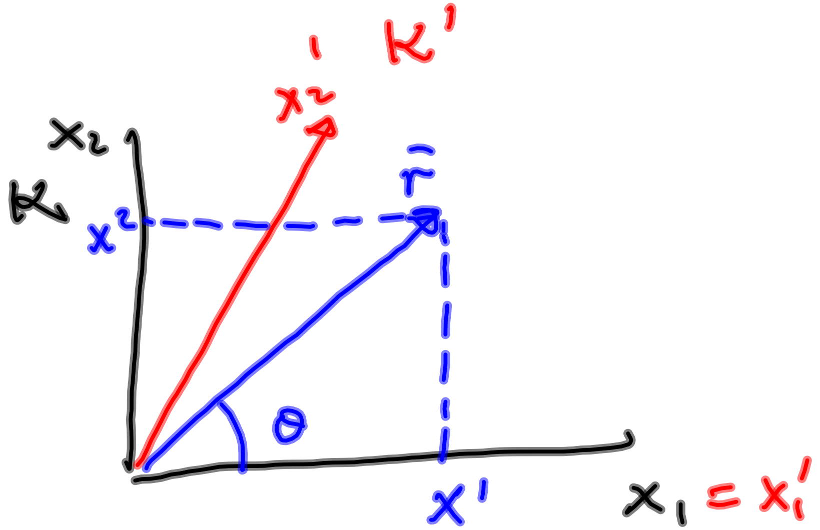
Next time we are going to consider the following transformation:



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