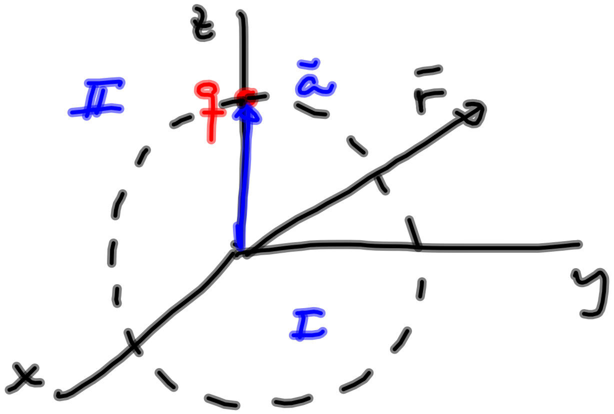
Applications of $\nabla^2\phi=0$ with spherical symmetry. 11/1

1) Find the potential of a charge q at a distance $\underline{a}$ from the origin in terms of Legendre polynomials.



- Orient the z axis so that it is // to $\overline{o-a}$
- $\nabla^2\phi=0$ everywhere except for $\bar{r}=(0,0,a)$.
- Devide space in two regions where $\nabla^2\phi=0$.

I : $r \leq a$

$$\phi^{I}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell \, r^\ell \, P_\ell(\cos\theta)$$

$B_\ell = 0$ since $r^{-(\ell+1)}$ diverges at $r=0$

II : $r \geq a$

$$\phi^{II}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} \, P_\ell(\cos\theta)$$

(no positive powers of r because r^ℓ diverges as $r \to \infty$).

At $r = a$:

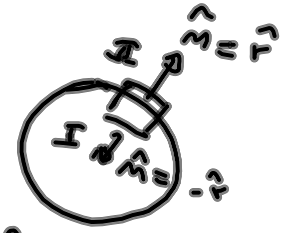
$$\phi^{I}(r=a,\theta) = \phi^{II}(r=a,\theta)$$

potential is continuous.

$\bar{E} = -\bar{\nabla}\phi$

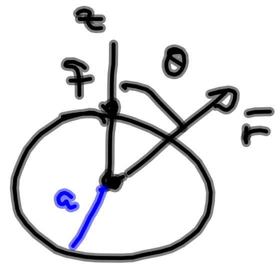
$\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\varepsilon_0}$

$E^{II}_r - E^{I}_r = \frac{\rho}{\varepsilon_0}$



$$-\frac{\partial \phi^{II}}{\partial r}\Big|_{r=a} + \frac{\partial \phi^{I}}{\partial r}\Big|_{r=a} = \frac{\rho}{\varepsilon_0}$$

$\rho = \sigma$ surface density of charge



$$\sigma = \frac{q\,\delta(\cos\theta - 1)}{2\pi a^2}$$

$$\int_{surf} \sigma\, dS = \int_{-1}^{1} q\, a^2 d(\cos\theta) \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \frac{\delta(\cos\theta - 1)}{2\pi a^2} =$$

$$= \frac{q\,\cancel{a^2}}{\cancel{a^2}} \underbrace{\int_{-1}^{1} \delta(\cos\theta - 1)}_{1} = q$$

Since $\phi^{I}\big|_a = \phi^{II}\big|_a$

$$A_\ell \, a^\ell = \frac{B_\ell}{a^{\ell+1}} \Rightarrow \boxed{A_\ell = \frac{B_\ell}{a^{2\ell+1}}}$$

and:

$$-\frac{\partial \phi^{II}}{\partial r}\Big|_a + \frac{\partial \phi^{I}}{\partial r}\Big|_a = \frac{q\,\delta(\cos\theta - 1)}{2\pi a^2}$$

$$\sum_{\ell=0}^{\infty}\left[(\ell+1)\frac{B_\ell}{a^{\ell+2}} + \frac{\ell\, B_\ell}{a^{\ell+2}}\right] P_\ell(\cos\theta) =$$

$$= \sum_{\ell=0}^{\infty} (2\ell+1)\frac{B_\ell}{a^{\ell+2}} P_\ell(\cos\theta) = q\frac{\delta(\cos\theta-1)}{2\pi a^2} \qquad (1)$$

I'll multiply both sides of ① by $P_{\ell'}(\cos\theta)$, and integrate over $\cos\theta$ in $[-1,1]$:

$$\sum_{\ell=0}^{\infty}(2\ell+1)\frac{B_\ell}{a^{\ell+2}}\underbrace{\int_{-1}^{1}P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\,d(\cos\theta)}_{\frac{2}{2\ell'+1}\delta_{\ell,\ell'}} =$$

$$= \frac{q}{2\pi a^2\varepsilon_0}\underbrace{\int_{-1}^{1}P_{\ell'}(\cos\theta)\,\delta(\cos\theta-1)\,d(\cos\theta)}_{P_{\ell'}(1)=1}$$

$$\frac{2B_{\ell'}}{a^{\ell'+2}} = \frac{q}{2\pi a^2\varepsilon_0} \Rightarrow \boxed{B_{\ell'} = \frac{q\,a^{\ell'}}{4\pi\varepsilon_0}} \text{ and } \boxed{A_{\ell'} = \frac{q}{4\pi\varepsilon_0 a^{\ell'+1}}}$$

Then:

$$\phi^{I}(r,\theta) = \frac{q}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\frac{r^{\ell}}{a^{\ell+1}}P_\ell(\cos\theta)$$

$$\phi^{II}(r,\theta) = \frac{q}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\frac{a^{\ell}}{r^{\ell+1}}P_\ell(\cos\theta)$$

or

2

$$\boxed{\phi(r,\theta) = \frac{q}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\theta)}$$

correct for $\vec{a} = (0,0,a)$ $a>0$. If $a<0$ then $P_\ell(-\cos\theta) = (-1)^\ell P_\ell(\cos\theta)$

Where $r_<$ ($r_>$) is the smaller (larger) between a and r.

Prove it using $\sigma = \frac{q\,\delta(\cos\theta+1)}{2\pi a^2}$

We know that

$$\phi_q = \frac{q}{4\pi\varepsilon_0} \frac{1}{|\bar{r} - \bar{a}|} \qquad 3$$

then comparing 3 with 2 we find that:

$$\boxed{\frac{1}{|\bar{r} - \bar{a}|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta)}$$

Use in homework together with principle of superposition.

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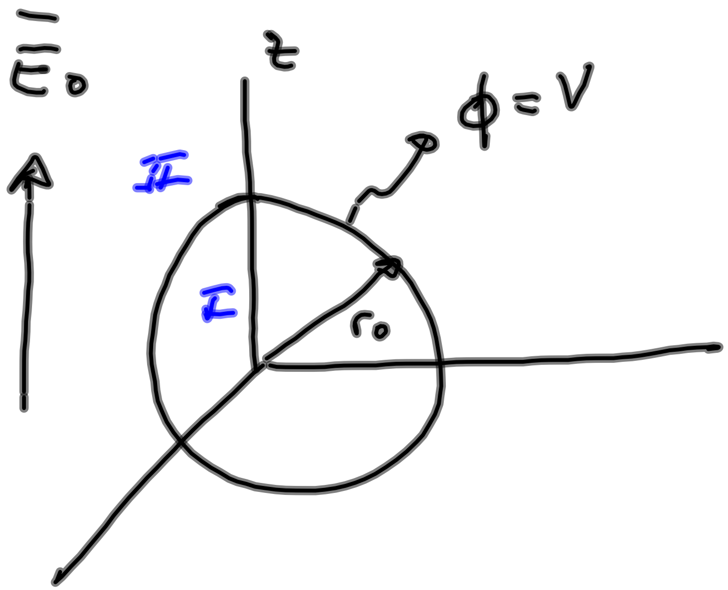
Application: Problem 12.3.13, a (Book).



$\phi(\bar{r}) = ?$

and find $\sigma(\theta, \varphi)$ on the sphere.

- Two regions because $\nabla^2\phi = 0$ not valid at $r = r_0$.
- Orient $\hat{z} \parallel \bar{E}_0$ to have azimuthal symmetry.

I: $r \leq r_0$

$$\phi^{I}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

II : $r \geq r_0$

$\bar{E}_0$ arises from $\phi_0 = -E_0 z$

but $z = r\cos\theta = r\,P_1(\cos\theta)$

then

$$\phi^{II}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) - E_0 r\,P_1(\cos\theta)$$

At $r=a$ $\quad \phi^{I}\big|_a = \phi^{II}\big|_a = V$

$$-\frac{\partial\phi^{II}}{\partial r}\Big|_a + \frac{\partial\phi^{I}}{\partial r}\Big|_a = \frac{\sigma(\theta)}{\varepsilon_0}$$

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Since

$$\phi^{I}\big|_{r=a} = V$$

$$\sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta) = V = V P_0(\cos\theta)$$

Then using orthogonality $A_0 a^0 = V \therefore \boxed{A_0 = V}$

and $A_\ell = 0$ for $\ell \neq 0$.

Then we obtain:

$$\sum_{\ell=0}^{\infty} \frac{B_\ell}{a^{\ell+1}} P_\ell(\cos\theta) - E_0 a P_1(\cos\theta) = V P_0(\cos\theta)$$

Since P_0 and P_1 appear explicitely $\ell=0$ and 1 need to be considered separately:

For $\ell = 0$:

$$\frac{B_0}{a} = V \quad \therefore \quad \boxed{B_0 = Va}$$

For $\ell = 1$:

$$\frac{B_1}{a^2} - E_0 a = 0 \Rightarrow \boxed{B_1 = E_0 a^3}$$

For $\ell > 1$:

$$\boxed{B_\ell = 0}$$

$$\phi^{I}(r,\theta) = V \quad \text{and} \quad \phi^{II}(r,\theta) = \frac{aV}{r} + \frac{E_0 a^3}{r^2}\overbrace{P_1(\cos\theta)}^{\cos\theta} - E_0 r \overbrace{P_1(\cos\theta)}^{\cos\theta}$$

Now

$$-\left.\frac{\partial \phi^{II}}{\partial r}\right|_a + \underbrace{\left.\frac{\partial \phi^{I}}{\partial r}\right|_a}_{0} = \frac{\sigma(\theta)}{\varepsilon_0}$$

$$\left.\frac{aV}{r^2}\right|_{r=a} + \left.\frac{2E_0 a^3}{r^3}\cos\theta\right|_{r=a} + \left.E_0\cos\theta\right|_{r=a} =$$

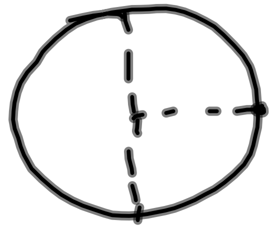
$$= \frac{V}{a} + 2E_0\cos\theta + E_0\cos\theta = \frac{V}{a} + 3E_0\cos\theta =$$

$$= \sigma(\theta)/\varepsilon_0$$

$$\boxed{\sigma(\theta) = \frac{V\varepsilon_0}{a} + 3\varepsilon_0 E_0\cos\theta}$$

If the sphere is grounded $V=0$ and

$\sigma(\cos\theta) = 3\,\varepsilon_0\,E_0 \cos\theta$



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Problems without azimuthal symmetry

Now $m \neq 0$ in the solution of $\nabla^2 \phi = 0$.

Then we need to solve the associated Legendre d.e. with $m \neq 0$.

The solutions are the associated Legendre Polynomials:

$P_\ell^m(\cos\theta)$

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$$P_\ell^m(x) = \underbrace{(-1)^m} (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)$$

Some authors put this $(-1)^m$ in the definition of $Y_\ell^m(\theta,\varphi)$ that we are going to see next.

Properties:

$-\ell \leq m \leq \ell$

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

$P_\ell^m(\cos\theta)$ are orthogonal in ℓ in $[-1,1]$:

$$\int_{-1}^{1} P_{\ell'}^m(x)\, P_\ell^m(x)\, dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell,\ell'}$$

The angular part of the solution to $\nabla^2\phi = 0$ is given by $P(\cos\theta)\, Q(\varphi) \propto P_\ell^m(\cos\theta)\, e^{\pm im\varphi}$

$-\ell \leq m \leq \ell$.

Spherical Harmonics

Let's define:

$$Y_\ell^m(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\varphi}$$

the book puts $(-1)^m$ here but we already have it in $P_\ell^m(\cos\theta)$.

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- ℓ are integers from 0 to ∞.
- m ranges from $-\ell$ to ℓ (integers).

Also:

$$Y_\ell^{-m}(\theta, \varphi) = (-1)^m \left(Y_\ell^m\right)^* (\theta, \varphi)$$

$\{ Y_\ell^m \}$ form an orthonormal basis in the interval $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \; Y_{\ell'}^{m'*} Y_\ell^m = \delta_{\ell\ell'} \delta_{mm'}$$

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So the general solution of $\nabla^2\phi = 0$ is:

$$\phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell,m}\, r^{\ell} + \frac{B_{\ell,m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta,\varphi)$$

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