11/13

Second Midterm: Thursday 11/15. (In class)

- Part II: due on Tuesday 11/20 (in class).
  (no homework due on 11/20).

- Bring your calculator.
- Provide numerical values when asked.

Review:
- metric tensor

$$x_i = g_{ij} x^j$$

$$x^i = g^{ij} x_j$$

- Minkowski space $\longrightarrow$ cont.

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$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \qquad \begin{array}{l} \mu = 0,1,2,3 \\ \nu = 0,1,2,3 \end{array}$$

$x_0 = ct \quad x_1 = x \quad x_2 = y \quad x_3 = z$

$$A_\mu A^\mu = g_{\mu\nu} A^\nu A^\mu = (A^0)^2 - |\bar{A}|^2$$

$$A^\mu = (A^0, A^x, A^y, A^z) = (A^0, \bar{A}).$$

- Tensor notation (demonstrations):

$r = (x_j x^j)^{1/2}$

Calculate using tensor notation $\frac{\partial r}{\partial x^i} = ?$

$$\frac{\partial r}{\partial x^i} = B_i \qquad \text{covariant tensor of rank 1.}$$

$$\boxed{B_i} = \partial_i r = \partial_i (x_k x^k)^{1/2} = \frac{1}{2} \frac{(\partial_i x_k) x^k + x_k \underbrace{\partial_i x^k}_{\delta_i{}^k}}{(x_k x^k)^{1/2}}$$

$$= \frac{1}{2} \frac{(\partial_i g_{k\ell} x^\ell) x^k + x_k \delta_i{}^k}{r} =$$

$$= \frac{1}{2r} (g_{k\ell} \underbrace{\partial_i x^\ell}_{\delta_i{}^\ell} x^k + x_i) =$$

$$= \frac{1}{2r} (\overbrace{g_{ki} x^k}^{x_i} + x_i) = \frac{1}{2r} 2x_i = \boxed{\frac{x_i}{r}} = m_i$$

$$\hat{m} = \frac{\bar{x}}{r} \quad \text{unit vector parallel to } \bar{x}.$$

- $\bar{\nabla} \times \bar{A}$ in tensor notation

$\bar{c} = \bar{a} \times \bar{b} \longrightarrow c_i = \varepsilon_{ijk}\, a^j b^k$

$\bar{\nabla} \times \bar{A} \longrightarrow \varepsilon_{ijk}\, \partial^j A^k$

metric tensor lowers or raises indices if needed.

- Separation of variables : cartesian and spherical.

To find coefficients be careful - Some times you need to consider separately certain values of ℓ, or m (in spherical) or m or n in cartesian

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See Example 12.3.13. a on 11/1 lecture.

$\ell = 0$

$\ell = 1$

$\ell > 1$

consider separately.

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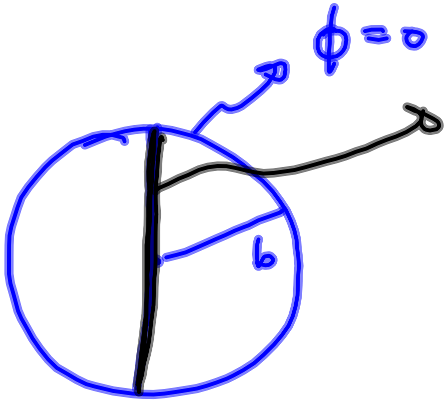
Applications of Green functions:

1)



$$\rho(\bar{r}') = \frac{Q}{2b}\,\frac{1}{2\pi r'^2}\left[\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1)\right]$$

$$\int_V \rho(\bar{r}')\, d^3r' = Q$$

Find $\phi(\bar{r})$ for $r \leq b$.

$$\phi(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_V G(\bar{r},\bar{r}')\,\rho(\bar{r}')\, d^3r'$$

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$G(\bar{r},\bar{r}')$ can be expanded in terms of $Y_{\ell m}$ as you did in the homework then:

$$\phi(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_0^b r'^2 dr' \int_{-1}^{1} d(\cos\theta') \int_0^{2\pi} d\varphi' \, \rho(\bar{r}') G(\bar{r},\bar{r}') =$$

$$= \frac{Q}{2\pi \, 2b} \frac{4\pi}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} Y_{\ell m}(\theta,\varphi)$$

$$\int_0^b \left[\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{r^{\ell} r'^{\ell}}{b^{2\ell+1}} \right] dr' \int_{-1}^{1} d(\cos\theta') \left[\delta(\cos\theta'+1) + \right.$$

$$\left. + \delta(\cos\theta'-1) \right] \int_0^{2\pi} d\varphi' \, Y_\ell^{-m}(\theta',\varphi')$$

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φ' dependence : all the dependance is in $Y_\ell^{-m}(\theta,\varphi')$

$$\int_0^{2\pi} d\varphi' \, \underbrace{Y_\ell^{-m}(\theta',\varphi')} = 2\pi \sqrt{\frac{2\ell+1}{4\pi}} \, P_\ell(\cos\theta') \, \delta_{m,0}$$

$$Y_\ell^{-m}(\theta',\varphi') \propto P_\ell(\cos\theta') \, e^{-im\varphi'}$$

$$\int_0^{2\pi} e^{-im\varphi'} d\varphi' = 2\pi \, \delta_{m,0}$$

Then

$$\phi(\bar{r}) = \frac{Q}{4\pi\, 2b\, \varepsilon_0} \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \int_0^b \left[\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{r^{\ell} r'^{\ell}}{b^{2\ell+1}} \right] dr'$$

$$\underbrace{\left(P_\ell(1) + P_\ell(-1) \right)}_{\text{from } \int d\cos\theta'} =$$

2 for ℓ even
0 for ℓ odd

$$= \frac{Q}{4\pi b\, \varepsilon_0} \sum_{j=0}^{\infty} P_{2j}(\cos\theta) \int_0^b \left[\frac{r_<^{2j}}{r_>^{2j+1}} - \frac{r^{2j} r'^{2j}}{b^{4j+1}} \right] dr' =$$

$$= \frac{Q}{4\pi b \varepsilon_0} \sum_{j=0}^{\infty} P_{2j}(\cos\theta) \left[\int_0^r \frac{dr'\, r'^{2j}}{r^{2j+1}} + \int_r^b \frac{dr'\, r^{2j}}{r'^{2j+1}} - \frac{r^{2j}}{b^{4j+1}} \int_0^b r'^{2j} dr' \right]$$

$$= \frac{Q}{4\pi b \varepsilon_0} \left\{ \sum_{j \geq 0}^{\infty} P_{2j}(\cos\theta) \left[\frac{1}{2j+1} \frac{r'^{2j+1}}{r^{2j+1}} \Big|_0^r + \right.\right.$$

$$\left.\left. - \underbrace{\frac{r^{2j}}{2j} \frac{1}{r'^{2j}} \Big|_r^b}_{\text{for } j \neq 0} - \frac{r^{2j}}{b^{4j+1}} \frac{1}{2j+1} r'^{2j+1} \Big|_0^b \right] \right\} =$$

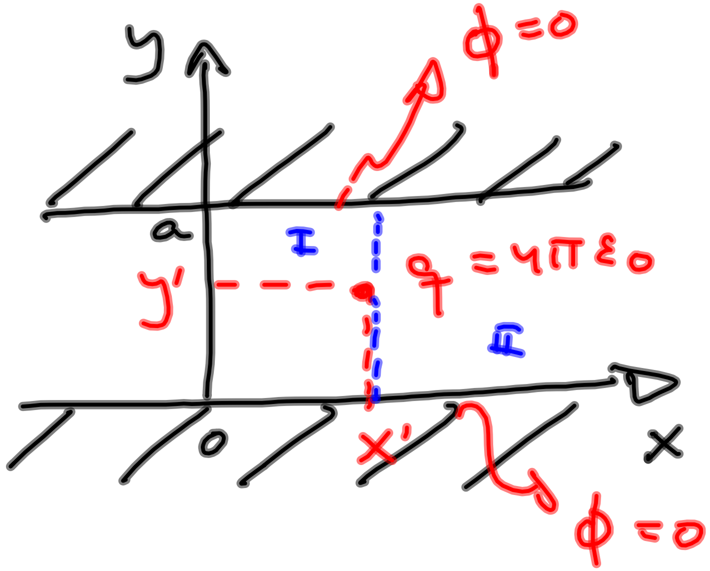
or $\ln r' \big|_r^b$

for $j = 0$

$$= \frac{Q}{2\pi b \varepsilon_0} \left\{ \left[\sum_{j=1}^{\infty} P_{2j}(\cos\theta) \left[\frac{1}{(2j+1)} - \frac{r^{2j}}{2j\, b^{2j}} + \frac{1}{2j} - \right.\right.\right.$$

$$\left.\left. - \frac{r^{2j}}{b^{2j}} \frac{1}{2j+1} \right] + \ln b/r \right\}$$

Another example (for homework):



Find $G(\bar{r}, \bar{r}')$ for this geometry.

$$G(\bar{r}, \bar{r}') = G(x, y; x', y')$$

$G(\bar{r}, \bar{r}')$ is the potential at $\bar{r}$ of a charge $q = 4\pi\varepsilon_0$ placed at $\bar{r}'$ (inside V) with $\phi = 0$ on the volume surface.

$$\phi^{I} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{a}\, e^{\frac{n\pi x}{a}} \qquad x \le x'$$

$$\phi^{II} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi y}{a}\, e^{-\frac{n\pi x}{a}} \qquad x \ge x'$$

Use b.c. at $x = x'$

$$\phi^{I}(x=x', y, x', y') = \phi^{II}(x=x', y, x', y')$$

$$-\frac{\partial \phi^{II}}{\partial x}\bigg|_{x=x'} + \frac{\partial \phi^{I}}{\partial x}\bigg|_{x=x'} = \frac{\sigma(y, y')}{\varepsilon_0} = \frac{q\,\delta(y-y')}{\varepsilon_0}$$

Then A_n and B_n will be functions of x' and y'

Providing $G(x, x', y, y')$

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Result:

$$G(x, x', y, y') = 4 \sum_{n=1}^{\infty} \frac{e^{\frac{n\pi}{a}(x_< - x_>)}}{n} \sin \frac{n\pi y}{a} \sin \frac{n\pi y'}{a}$$

$x_<$ and $x_>$ are the smaller and larger between x and x'.

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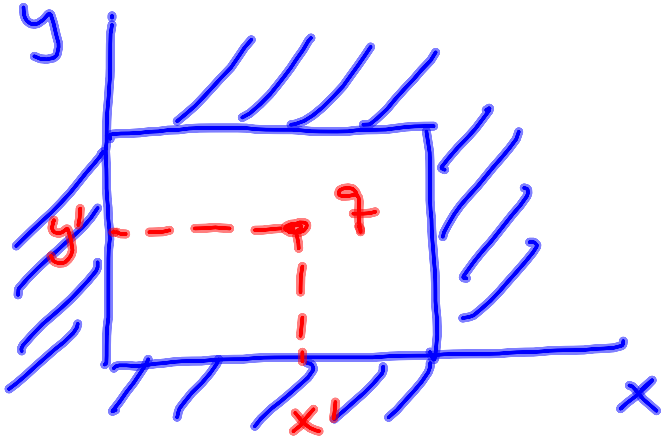
One more example:



$q = 2\pi \varepsilon_0$

Find potential at $\bar{r}$ due to q at r' to obtain

$G(x, y, x', y')$.

Calculate $G(x, x', y, y')$

Here one can chose in what direction to use sin or exp.

Exchange $x - y$, $x' - y'$: one goes from one case to the other.

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